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THE QUARTERLY JOURNAL OF MATHEMATICS

OXFORD SECOND SERIES

Edited by T. W. CHAUNDY, U. S. HASLAM-JONES, E. C. THOMPSON

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A PROPERTY OF SIMPLY TRANSITIVE GROUPS

By G. TEMPLE (*Oxford*)

[Received March 1960]

1. Transitive transformation groups

A CONTINUOUS transformation group is specified in terms of an open set of points V in the m-dimensional euclidean space of the variables $x = (x_1, x_2, \ldots, x_m)$, and of an open set of points P in the n-dimensional euclidean space of the parameters $a = (a_1, a_2, \ldots, a_n)$. We consider a set of transformations

$$x \to y\colon\ y = f(x; a), \tag{1}$$

each one of which is specified by a point a of P and maps the region V on to itself. We require this set of transformations to form a group G but we do not assume the functions f to be analytic or even differentiable.

If the group G is simply transitive, then there is a unique transformation, specified by a point b of P, which carries a prescribed point z of V into another prescribed point w of V, so that

$$w = f(z; b).$$

This implies that the dimensions of V and P are the same, i.e. that

$$m = n.$$

2. Canonical parameters

The discussion of transitive transformation groups is simplified if we introduce a certain *canonical* set of parameters as follows. Let ω be any arbitrary point of V, fixed once and for all. Then to any point a of P there corresponds a unique point α of V such that

$$\alpha = f(\omega; a).$$

Also, since the group is simply transitive, the point a of P is uniquely determined by α (and ω). The coordinates $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ of α are the 'canonical' parameters of the group G (for the base point ω).

To express the transformations of G in terms of the canonical parameters we eliminate the coordinates of a between the equations

$$y = f(x; a), \qquad \alpha = f(\omega; a), \tag{2}$$

thus obtaining the transformations

$$y = \phi(x; \alpha). \tag{2}'$$

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These transformations (2)′ form a continuous transitive group H (isomorphic with G), for which the variables x and the parameters α belong to the *same* open set of points.

3. The base point as variable and as parameter

Put $x = \omega$ in (2) and (2)′. Then

$$y = f(\omega; a), \qquad \alpha = f(\omega, a), \qquad y = \phi(\omega; \alpha).$$

Hence
$$\alpha = y = \phi(\omega; \alpha). \tag{3}$$

Thus the transformation specified by a canonical parameter α carries the base point ω into the point α.

Also, since the transformations (2)′ form a group, there is an identical transformation, with parameter c say, such that

$$x = f(x; c), \qquad \omega = f(\omega; c), \qquad x = \phi(x; \omega). \tag{4}$$

Thus the transformation specified by the canonical parameter ω is the identical transformation.

4. The parameter groups of H

The transformations of H form a group. Hence, if

$$y = \phi(x; \alpha), \qquad z = \phi(y; \beta),$$

then there exists a parameter γ such that

$$z = \phi(x; \gamma)$$

and γ is a function of α and β. Now put $x = \omega$ in the equations

$$y = \phi(x; \alpha), \quad \phi(x; \gamma) = z = \phi(y; \beta). \tag{5}$$

Then
$$y = \phi(\omega; \alpha) = \alpha, \quad \text{by (3)},$$

and
$$\phi(y; \beta) = \phi(\omega; \gamma) = \gamma, \quad \text{by (3)}.$$

Hence
$$\gamma = \phi(\alpha; \beta). \tag{6}$$

We can now prove our main theorem: *The set of transformations* H^*

$$y = \phi(\alpha; x)$$

(obtained by interchanging the variables and parameters in the transformation $y = \phi(x; \alpha)$ *of* H*) also form a group.*

For, if we write $x = \mu$, $\alpha = \lambda$, and $\beta = \xi$ in equations (5) and (6), i.e. in

$$\phi\{x; \phi(\alpha, \beta)\} = \phi\{\phi(x; \alpha); \beta\},$$

we obtain
$$\phi\{\mu; \phi(\lambda; \xi)\} = \phi\{\phi(\mu; \lambda); \xi\}.$$

Hence, the product of two transformations of H^*,

$$\eta = \phi(\lambda; \xi), \qquad \xi = \phi(\mu; \eta)$$

with parameters λ and μ, is the transformation

$$\xi = \phi\{\mu; \phi(\lambda; \xi)\} = \phi\{\phi(\mu; \lambda); \xi\} = \phi(\nu; \xi),$$

where $\nu = \phi(\mu; \lambda)$,

i.e. another transformation of H^* with parameter ν. Similarly, the inverse of any transformation of H^* exists and belongs to H^*. Also, by (3),

$$\xi = \phi(\omega; \xi),$$

so that the transformation with parameter ω furnishes the identical transformation in H^*.

Hence the set of transformations H^* forms a group, which is clearly simply transitive, and which might be called the 'transpose' of H.

This result was obtained by E. Vessiot (1) in the course of a study of non-linear differential equations, by examining the differential equations satisfied by the functions $f(x; a)$ of (1).

5. Illustrations

(1) A simple example, suggested by Professor G. Higman, is provided by the group

$$y_1 = a_1 x_1, \qquad y_2 = a_1 x_2 + a_2,$$

where the variables and parameters lie in the open sets

$$x_1 > 0, \qquad y_1 > 0, \qquad a_1 > 0.$$

The parameters a_1, a_2 are canonical for the base point $(1, 0)$. The transposed group is

$$y_1 = a_1 x_1, \qquad y_2 = a_2 x_1 + x_2.$$

(2) If G is the direct product of rotations and magnifications in a plane, then

$$y_1 = a_1(x_1 \cos a_2 - x_2 \sin a_2), \qquad y_2 = a_1(x_1 \sin a_2 + x_2 \cos a_2).$$

The canonical parameters for the base point $(1, 0)$ are given by

$$\alpha_1 \pm i\alpha_2 = a_1 e^{\pm i a_2},$$

and the group H is given by

$$y_1 \pm i y_2 = (\alpha_1 \pm i\alpha_2)(x_1 \pm i x_2),$$

and is identical with its transpose H^*.

(3) In the group

$$y_1 = x_1 + a_1 e^{-x_2}, \qquad y_2 = x_2 + a_2,$$

the parameters a_1, a_2 are canonical for the base point $(0, 0)$, and the transposed group is

$$y_1 = x_1 e^{-a_2} + a_1, \qquad y_2 = x_2 + a_2.$$

(4) If $y = x_k(t)$ $(k = 1, 2, 3)$ are any three solutions of a Riccati equation of the form

$$\frac{dy}{dt} = P(t) + Q(t)y + R(t)y^2,$$

then, in general, any other solution $y = x(t)$ of this equation is given by the formula

$$\frac{x-x_3}{x-x_1} \cdot \frac{x_2-x_1}{x_2-x_3} = \text{a constant}.$$

Hence the set of transformations

$$x_k \to y_k, \qquad \frac{y_k-x_3}{y_k-x_1} = a_k \frac{x_2-x_3}{x_2-x_1} \quad (k = 1, 2, 3)$$

forms a group. The canonical parameters $(\alpha_1, \alpha_2, \alpha_3)$ for the base point $(\omega_1, \omega_2, \omega_3)$ are given by

$$\frac{\alpha_k-\omega_3}{\alpha_k-\omega_1} = a_k \frac{\omega_2-\omega_3}{\omega_2-\omega_1},$$

whence the equations of the group H are

$$\frac{y_k-x_3}{y_k-x_1} = \frac{\alpha_k-\omega_3}{\alpha_k-\omega_1} \cdot \frac{\omega_2-\omega_1}{\omega_2-\omega_3} \cdot \frac{x_2-x_3}{x_2-x_1},$$

and the transposed group H^* is

$$\frac{y_k-\alpha_3}{y_k-\alpha_1} = \frac{\alpha_2-\alpha_3}{\alpha_2-\alpha_1} \cdot \frac{\omega_2-\omega_1}{\omega_2-\omega_3} \cdot \frac{x_k-\omega_3}{x_k-\omega_1},$$

which is clearly the direct product of three one-dimensional projective groups.

[This paper was written in the course of research sponsored by the Office of Ordnance Research, U.S. Army.]

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FOURIER TRANSFORMATION OF THE n-DIMENSIONAL RADIAL DELTA FUNCTION

By S. VEMBU (*Madras*)

[Received 7 September 1960]

1. Introduction

THIS paper is concerned with the Fourier transform of an 'n-dimensional' delta function, i.e. of the function $\delta(|\mathbf{r}|^2-l^2)$, which has a non-zero value only on the surface of a hypersphere of radius l. The Fourier transform is shown to be connected with the Bessel function $J_{\frac{1}{2}(n-2)}(x)$, where n is the dimension of the hyperspace. This Fourier transform is also found to be closely related to the Fourier transform of a sphere of uniform density in a space of dimension $n-2$. Some particular results have also been worked out for $n = 1, 2, 3, 4$. (Here $|\mathbf{r}|$ is the magnitude of an n-vector which is also equal to r, the radial component of the spherical polar coordinates representing the terminus of the n-vector.)

2. Normalization of the delta function

The delta function can be normalized by taking its integral over the whole of the n-dimensional space to be unity. The normalization constant K is therefore given by the equation

$$K \int \dots \int \delta(|\mathbf{r}|^2-l^2)\, d\tau = 1. \tag{2.1}$$

The volume element of an n-dimensional space is given in terms of spherical polar coordinates by

$$d\tau = r^{n-1} \sin^{n-2}\theta \sin^{n-3}\phi_1 \dots \sin\phi_{n-3}\, dr d\theta d\phi_1 \dots d\phi_{n-2}, \tag{2.2}$$

where θ, $\phi_1, \dots, \phi_{n-3}$ are half-turn angles with the range $(0, \pi)$, while ϕ_{n-2} is a full-turn angle with integration limits $(0, 2\pi)$.

Since $\delta(|\mathbf{r}|^2-l^2) \equiv \delta(r^2-l^2)$ is a function only of r, the integral in equation (2.1) is

$$\int_0^\infty r^{n-1}\delta(|\mathbf{r}|^2-l^2)\, dr \int_0^\pi \sin^{n-2}\theta\, d\theta \int_0^\pi \sin^{n-3}\phi_1\, d\phi_1 \dots \int_0^{2\pi} d\phi_{n-2}. \tag{2.3}$$

The integral over the ϕ can be shown to be given by

$$C(\phi) = 2\pi^{\frac{1}{2}n-\frac{1}{2}}/\Gamma(\tfrac{1}{2}n-\tfrac{1}{2}) \tag{2.4}$$

and, including the integral over θ, the value is

$$s_n = C(\phi) \int_0^{\pi} \sin^{n-2}\theta \, d\theta = 2\pi^{\frac{1}{2}n}/\Gamma(\tfrac{1}{2}n). \tag{2.5}$$

Thus we obtain from equation (2.1)

$$K = \Gamma(\tfrac{1}{2}n)/\pi^{\frac{1}{2}n} l^{n-2}. \tag{2.6}$$

3. The Fourier transform of the normalized delta function

We define the Fourier transform of a function $f(\mathbf{r})$ as

$$\phi(\boldsymbol{\alpha}) = \int \ldots \int f(\mathbf{r}) \exp(2\pi i \mathbf{r} . \boldsymbol{\alpha}) \, d\tau, \tag{3.1}$$

where $\boldsymbol{\alpha}$ is again a vector in an n-dimensional space (the so-called *Fourier* or *reciprocal* space). For the δ-function this becomes

$$K \int \ldots \int \delta(r^2 - l^2) \exp(2\pi i \alpha r \cos\theta) \, d\tau. \tag{3.2}$$

Integrating over the azimuthal angles $\phi_1, \ldots, \phi_{n-2}$ and r we get the transform as

$$\{\tfrac{1}{2} K l^{n-2} C(\phi)\} \int_0^{\pi} \exp(2\pi i \alpha l \cos\theta) \sin^{n-2}\theta \, d\theta. \tag{3.3}$$

Substituting for K and $C(\phi)$ we have

$$\{\Gamma(\tfrac{1}{2}n)/\sqrt{\pi}\,\Gamma(\tfrac{1}{2}n - \tfrac{1}{2})\} \int_0^{\pi} \exp(2\pi i \alpha l \cos\theta) \sin^{n-2}\theta \, d\theta, \tag{3.4}$$

which further reduces to

$$\{2\Gamma(\tfrac{1}{2}n)/\sqrt{\pi}\,\Gamma(\tfrac{1}{2}n - \tfrac{1}{2})\} \int_0^{\frac{1}{2}\pi} \cos(2\pi\alpha l \cos\theta) \sin^{n-2}\theta \, d\theta. \tag{3.5}$$

Now the Bessel function of order ν is defined by the integral

$$J_\nu(z) = \{2(\tfrac{1}{2}z)^\nu/\sqrt{\pi}\,\Gamma(\nu + \tfrac{1}{2})\} \int_0^{\frac{1}{2}\pi} \cos(z\cos\alpha) \sin^{2\nu}\alpha \, d\alpha. \tag{3.6}$$

Using this, we reduce the expression (3.5) for the Fourier transform of the delta function to

$$2^{\frac{1}{2}n-1} J_{\frac{1}{2}n-1}(2\pi\alpha l)/(2\pi\alpha l)^{\frac{1}{2}n-1}. \tag{3.7}$$

4. Fourier transform of a normalized sphere

The function $f(\mathbf{r})$ describing a sphere of uniform density is given by

$$\left.\begin{aligned} f(\mathbf{r}) &= \rho \quad (|r| \leqslant l) \\ f(\mathbf{r}) &= 0 \quad (|r| > l) \end{aligned}\right\}. \tag{4.1}$$

The normalization condition is

$$\int f(\mathbf{r})\,d\tau = 1,$$

which gives
$$\rho s_n \int_0^l r^{n-1}\,dr = 2\rho\pi^{\frac{1}{2}n} l^n / n\Gamma(\tfrac{1}{2}n) = 1, \tag{4.2}$$

so that
$$\rho = \Gamma(\tfrac{1}{2}n+1)/(\pi l^2)^{\frac{1}{2}n}. \tag{4.3}$$

The Fourier transform of this sphere is

$$\rho \int_0^l r^{n-1}\,dr\, C(\phi) \int_0^\pi \exp(2\pi i\alpha r\cos\theta)\sin^{n-2}\theta\,d\theta \tag{4.4}$$

$$= 2\rho C(\phi) \int_0^l r^{n-1}\,dr \int_0^{\frac{1}{2}\pi} \cos(2\pi\alpha r\cos\theta)\sin^{n-2}\theta\,d\theta \tag{4.5}$$

$$= 2^{\frac{1}{2}n}\sqrt{\pi}\, l^n \rho J_{\frac{1}{2}n}(2\pi\alpha l)/(2\pi\alpha l)^{\frac{1}{2}n}. \tag{4.6}$$

Substituting for ρ from (4.3) we have the Fourier transform in the form

$$2^{\frac{1}{2}n}\Gamma(\tfrac{1}{2}n+1) J_{\frac{1}{2}n}(2\pi\alpha l)/(2\pi\alpha l)^{\frac{1}{2}n}. \tag{4.7}$$

5. Results

Equations (3.7) and (4.7) can be written in a slightly more elegant form by making the substitutions

$$x = 2\pi\alpha l, \qquad p = \tfrac{1}{2}n-1, \qquad \Gamma(\tfrac{1}{2}n) = p! \tag{5.1}$$

Then the normalized delta function takes the form

$$p!\,\delta(r^2-l^2)/\pi(\pi l^2)^p, \tag{5.2}$$

and its Fourier transform is (1)

$$\{2^p p!\, J_p(x)/x^p\} \equiv \Lambda_p(x). \tag{5.3}$$

Similarly the density ρ of the normalized sphere becomes

$$\rho = (p+1)!/(\pi l^2)^{p+1}, \tag{5.4}$$

and its Fourier transform is

$$\{2^{p+1}(p+1)!\, J_{p+1}(x)/x^{p+1}\} \equiv \Lambda_{p+1}(x). \tag{5.5}$$

It is very interesting to note that the radial part of the Fourier transform of the normalized delta function in n dimensions is identical with the Fourier transform of the normalized uniform sphere in $n-2$ dimen-

sions. This will be clear from the particular cases of $n = 1, 2, 3, 4$, for which the Fourier transforms are as follows:

	Fourier transform of	
n	*Delta function*	*Uniform sphere*
1	$\cos(2\pi\alpha l)$	$\sin(2\pi\alpha l)/2\pi\alpha l$
2	$J_0(2\pi\alpha l)$	$2J_1(2\pi\alpha l)/2\pi\alpha l$
3	$\sin(2\pi\alpha l)/2\pi\alpha l$	$3\{\sin(2\pi\alpha l)/(2\pi\alpha l)^3 - -\cos(2\pi\alpha l)/(2\pi\alpha l)^2\}$
4	$2J_1(2\pi\alpha l)/2\pi\alpha l$	$8J_2(2\pi\alpha l)/2\pi\alpha l$

The results for $n = 1, 2, 3$ are known and used in diffraction theory. Because of the property of Bessel functions of half-odd integers, the Fourier transforms of delta functions are expressible in terms of trigonometric functions for odd values of n and similarly of uniform spheres, for even values of n. The extension to an n-dimensional sphere or spherical shell whose density is a function of r only is obvious.

The author expresses his indebtedness to Professor G. N. Ramachandran for suggesting this problem and for the lively discussion of this subject.

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SPIN REPRESENTATION OF A DIRECT SUM AND A DIRECT PRODUCT (II)

By A. O. MORRIS (*Aberystwyth*)

[Received 22 October 1960]

Introduction

THE spin representation of the rotation group gives a new group, known as the *covering* group (**2**), which is isomorphic to the rotation group, such that two elements of the covering group correspond to one element of the rotation group. In § 1 it is shown that, whereas the covering group of the rotation group is uniquely defined, when the results are extended to the full orthogonal group, there exist two such covering groups. In (**4**) the spin character of the direct sum and direct product of two orthogonal matrices of positive determinant were shown to be expressible in terms of the characters (ordinary and spin) of the two matrices. In § 2 and § 3, these results are extended to include orthogonal matrices of negative determinant.

1. A set of matrices M_r ($r = 1, 2, \ldots, n = 2\nu$) of degree 2^ν can be found which satisfy the relations (**1**, **2**, **3**)

$$\left.\begin{aligned} M_r^2 &= E \\ M_r M_s &= -M_s M_r \quad (r \neq s) \end{aligned}\right\}. \tag{1.1}$$

These matrices are shown to generate a group of order $2^{2\nu+1}$, the matrices forming a representation of degree 2^ν of the group. If $A = [a_{rs}]$ is an n-rowed orthogonal matrix, then the quantities

$$M'_r = \sum a_{rs} M_s$$

also satisfy the relations (1.1). Thus the 2ν quantities M'_r also generate a representation of degree 2^ν of the group. Since these two representations of the group must be equivalent, a matrix $T(A)$ must exist such that

$$M'_r = T(A)^{-1} M_r T(A).$$

The set of matrices $T(A)$ form a representation of degree 2^ν of the orthogonal group of degree $n = 2\nu$ and are unique except for a scalar factor λ. This scalar factor may be restricted to have the value ± 1, and hence we have a two-valued matrix representation $\pm T(A)$ of degree 2^ν of the orthogonal group. This is known as the *spin representation* of the orthogonal group.

Quart. J. Math. Oxford (2), 12 (1961), 169-76.

A minor modification in the argument shows that the group of matrices $\pm T(A)$ also form a spin representation of the orthogonal group of degree $n = 2\nu+1$.

Any real orthogonal matrix can be replaced by its equivalent complex diagonal form, which must take one of the four forms

$$A_1 = \text{diag}(e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_\nu}, e^{-i\theta_\nu}), \tag{1.21}$$

$$A_2 = \text{diag}(1, -1, e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_{\nu-1}}, e^{-i\theta_{\nu-1}}), \tag{1.22}$$

$$A_3 = \text{diag}(1, e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_\nu}, e^{-i\theta_\nu}), \tag{1.23}$$

$$A_4 = \text{diag}(-1, e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_\nu}, e^{-i\theta_\nu}). \tag{1.24}$$

If $\zeta(A)$ denotes the basic spin character of A, i.e. the character of A in the representation defined by the $T(A)$, then the basic spin characters of these matrices are (1)

$$\zeta(A_1) = \zeta(A_3) = \prod_{r=1}^{\nu} (2\cos\tfrac{1}{2}\theta_r), \tag{1.31}$$

$$\zeta(A_2) = 0, \tag{1.32}$$

$$\zeta(A_4) = \prod_{r=1}^{\nu} (2\sin\tfrac{1}{2}\theta_r). \tag{1.33}$$

Also, we have

$$\zeta'(A_1) = \prod_{r=1}^{\nu} (2i\sin\tfrac{1}{2}\theta_r), \tag{1.34}$$

where $\zeta'(A_1)$ denotes the difference spin character of the matrix A_1.

The group of matrices $\pm T(A)$ which form a spin representation of the orthogonal group of degree n is said to form a *covering group* of the orthogonal group. It is now shown that, whereas this covering group is uniquely defined for the rotation group, there are correspondingly two covering groups of the full orthogonal group.

Let S denote any orthogonal matrix of positive determinant and T any orthogonal matrix of negative determinant. Let the correspondence

$$S \to \overline{S}, -\overline{S}; \qquad T \to \overline{T}, -\overline{T}$$

give a basic spin representation of the full orthogonal group.

We prove the theorem:

THEOREM I. *The correspondence*

$$S \to \overline{S}, -\overline{S}; \qquad T \to i\overline{T}, -i\overline{T}$$

gives an alternative spin representation of the full orthogonal group.

The proof is evident. The product of two orthogonal matrices can

always be expressed in one of the four forms,

$$S_1S_2 = S_3, \quad S_1T_2 = T_3, \quad T_1S_2 = T_3, \quad T_1T_2 = S_3.$$

If a basic spin representation is given by

$$S_i \to \bar{S}_i, -\bar{S}_i; \qquad T_i \to \bar{T}_i, -\bar{T}_i,$$

then substitution in these equations shows that the representation

$$S_i \to \bar{S}_i, -\bar{S}_i; \qquad T_i \to i\bar{T}_i, -i\bar{T}_i$$

also gives a basic spin representation.

The group generated by the matrices $\bar{S}$, $i\bar{T}$ is a distinct group from the group generated by $\bar{S}$, $\bar{T}$. Thus, if T satisfies $T^2 = I$, then $\bar{T}$ must satisfy a minimal equation either $\bar{T}^2 = I$ or $\bar{T}^2 = -I$. Thus $\bar{T}$ will be a group element of order 2 or 4. But, if $\bar{T}$ is a group element of order 4, then $i\bar{T}$ is a group element of order 2, and conversely. The group generated by $\bar{S}$, $\bar{T}$ is called the *first* covering group, and the group generated by $\bar{S}$, $i\bar{T}$ is called the *second* covering group.

It is clear that in the development of Brauer and Weyl (**3**) there must have been at some point an implicit selection from two alternatives, and it is of interest to determine at which point this arises. For an odd-rowed matrix of negative determinant this is immediately apparent. They simply assign $\pm 2^\nu$ as the basic spin character of the matrix

$$E_n = \text{diag}\{(-1)^{2\nu+1}\},$$

while from the above argument we see that $\pm i2^\nu$ also give a basic spin character of E_n. Here, we assign $\pm i(2i)^\nu$ or $\pm(2i)^\nu$ as the basic spin character of E_n. This step will be justified in a future publication when the results of this paper are applied to the study of the spin representation of the symmetric group.

Thus the formula (1.33) is replaced by the two formulae

$$\zeta_1(A_4) = i\prod_{r=1}^{\nu}(2i\sin\tfrac{1}{2}\theta_r), \tag{1.33$'$}$$

$$\zeta_2(A_4) = \prod_{r=1}^{\nu}(2i\sin\tfrac{1}{2}\theta_r). \tag{1.33$''$}$$

2. In a previous paper (**4**) a value was found for the basic spin character of the direct sum of two orthogonal matrices of positive determinant. In this section, orthogonal matrices of negative determinant are also considered.

Since, by (1.32), the basic spin character of an even-rowed orthogonal matrix of negative determinant is zero, the only new combination which

need be considered is the direct sum of A_1 and A_4. The following theorem is proved:

THEOREM II. *If*

$$A = \operatorname{diag}(-1, e^{i\phi_1}, e^{-i\phi_1}, \ldots, e^{i\phi_\mu}, e^{-i\phi_\mu}),$$

$$B = \operatorname{diag}(e^{i\theta_1}, e^{-i\theta_1}, \ldots, e^{i\theta_\nu}, e^{-i\theta_\nu}),$$

then
$$\zeta_1(A \dot{+} B) = \zeta_1(A)\zeta'(B), \tag{2.11}$$

and
$$\zeta_2(A \dot{+} B) = \zeta_2(A)\zeta'(B), \tag{2.12}$$

where $\zeta'(B)$ is the basic difference spin character of B.

The proof is immediate. By the formulae (1.33)′ and (1.34), we obtain that

$$\zeta_1(A) = i \prod_{r=1}^{\mu} (2i \sin \tfrac{1}{2}\phi_r),$$

and
$$\zeta'(B) = \prod_{r=1}^{\nu} (2i \sin \tfrac{1}{2}\theta_r),$$

and
$$\zeta_1(A \dot{+} B) = i \prod_{r=1}^{\mu} (2i \sin \tfrac{1}{2}\phi_r) \prod_{r=1}^{\nu} (2i \sin \tfrac{1}{2}\theta_r).$$

Hence we have $\zeta_1(A \dot{+} B) = \zeta_1(A)\zeta'(B)$, proving (2.11). (2.12) is proved in the same way.

The extension to the case of the direct sum of more than two orthogonal matrices of positive determinant is trivial. An interesting result is obtained when the direct sum contains an odd number of orthogonal matrices of negative determinant. The following theorem is proved:

THEOREM III. (i) *If $A = \sum_{r=1}^{p} A_r$ denotes the direct sum of $p = 2\mu+1$ orthogonal matrices of the form A_4, then*

$$\zeta_\alpha(A) = (2i)^\mu \prod_{r=1}^{p} \zeta_\alpha(A_r). \tag{2.2}$$

(ii) *If*
$$A' = \sum_{r=1}^{p} A_r \dot{+} \sum_{s=p+1}^{p+q} B_s$$

denotes the direct sum of $p = 2\mu+1$ orthogonal matrices of the form A_4 and q orthogonal matrices of the form A_1, then

$$\zeta_\alpha(A') = (2i)^\mu \prod_{r=1}^{p} \zeta_\alpha(A_r) \prod_{s=p+1}^{p+q} \zeta'(B_s), \tag{2.3}$$

where $\alpha = 1$ or 2.

The theorem is proved by substituting formulae (1.33)′ or (1.33)″ and (1.34) in these equations.

3. To simplify the enunciation, in addition to the matrices A_1, A_2, A_3, A_4, we define the set of matrices

$$\left.\begin{aligned} B_1 &= \operatorname{diag}(e^{i\phi_1}, e^{-i\phi_1},\ldots, e^{i\phi_\mu}, e^{-i\phi_\mu}) \\ B_2 &= \operatorname{diag}(1, -1, e^{i\phi_1}, e^{-i\phi_1},\ldots, e^{i\phi_{\mu-1}}, e^{-i\phi_{\mu-1}}) \\ B_3 &= \operatorname{diag}(1, e^{i\phi_1}, e^{-i\phi_1},\ldots, e^{i\phi_\mu}, e^{-i\phi_\mu}) \\ B_4 &= \operatorname{diag}(-1, e^{i\phi_1}, e^{-i\phi_1},\ldots, e^{i\phi_\mu}, e^{-i\phi_\mu}) \end{aligned}\right\}. \tag{3.1}$$

The basic spin characters of the direct products $A_1\times B_1$, $A_1\times B_3$, and $A_3\times B_3$ have already been found (**4**). For the sake of completeness, these results are quoted in the enunciation of Theorem IV. Denote the characters of A_1, A_2, A_3, A_4 by $[\lambda]_{11}$, $[\lambda]_{12}$, $[\lambda]_{13}$, $[\lambda]_{14}$ respectively, and the characters of B_1, B_2, B_3, B_4 by $[\lambda]_{21}$, $[\lambda]_{22}$, $[\lambda]_{23}$, $[\lambda]_{24}$ respectively. In this paper primes are used to denote difference characters.

As a preliminary, we consider two simple examples:

(i) If $A = \operatorname{diag}(e^{i\theta}, e^{-i\theta})$ and $B = \operatorname{diag}(e^{i\phi}, e^{-i\phi})$ then

$$\begin{aligned} \zeta'(A\times B) &= 4\sin\tfrac{1}{2}(\theta+\phi)\sin\tfrac{1}{2}(\theta-\phi) \\ &= 2\cos\phi - 2\cos\theta. \end{aligned} \tag{3.11}$$

(ii) If $A = \operatorname{diag}(e^{i\theta}, e^{-i\theta})$ and $B = \operatorname{diag}(1, -1)$, then

$$\begin{aligned} \zeta(A\times B) &= 4\sin\tfrac{1}{2}\theta\cos\tfrac{1}{2}\theta \\ &= 2\sin\theta. \end{aligned} \tag{3.12}$$

The following results, which are stated in the form of a theorem, are proved in a similar manner to Theorems II, III, and IV of (**4**). Hence full details of the proof will not be given.

THEOREM IV. *If A_1, A_2, A_3, A_4, B_1, B_2, B_3, B_4 are the orthogonal matrices defined in* (1.2) *and* (3.1), *then*

(i) $\zeta(A_1\times B_1) = [\nu^\mu]_{21} +$
$\quad + \sum [\nu-\lambda_\mu, \nu-\lambda_{\mu-1},\ldots, \nu-\lambda_1]_{21}[\omega_1,\ldots,\omega_\nu]_{11}$,

(ii) $\zeta'(A_1\times B_1) = [\nu^\mu]_{21} +$
$\quad + \sum (-1)^{\Sigma\lambda_r}[\nu-\lambda_\mu, \nu-\lambda_{\mu-1},\ldots, \nu-\lambda_1]_{21}[\omega_1,\ldots,\omega_\nu]_{11}$,

(iii) $\zeta(A_1\times B_2) = [(\nu-1)^\mu]_{22}[1^\nu]'_{11} +$
$\quad + \sum [\nu-1-\lambda_\mu,\ldots, \nu-1-\lambda_1]_{22}[1+\omega_1,\ldots, 1+\omega_\nu]'_{11}$,

(iv) $\zeta'(A_1\times B_2) = [(\nu-1)^\mu]_{22}[1^\nu]'_{11} +$
$\quad + \sum (-1)^{\Sigma\lambda_r}[\nu-1-\lambda_\mu,\ldots, \nu-1-\lambda_1]_{22}[1+\omega_1,\ldots, 1+\omega_\nu]'_{11}$,

(v) $\zeta(A_3\times B_1) = [(\nu+\tfrac{1}{2})^\mu]_{21} +$
$\quad + \sum [\nu+\tfrac{1}{2}-\lambda_\mu,\ldots, \nu+\tfrac{1}{2}-\lambda_1]_{21}[\omega_1,\ldots,\omega_\nu]_{13}$,

(vi) $\zeta'(A_3\times B_1) = [(\nu+\frac{1}{2})^\mu]'_{21}+$
$+\sum(-1)^{\Sigma\lambda_r}[\nu+\frac{1}{2}-\lambda_\mu,\dots,\nu+\frac{1}{2}-\lambda_1]'_{21}[\omega_1,\dots,\omega_\nu]_{14}$,

(vii) $\zeta(A_4\times B_1) = [(\nu+\frac{1}{2})^\mu]'_{21}+$
$+\sum[\nu+\frac{1}{2}-\lambda_\mu,\dots,\nu+\frac{1}{2}-\lambda_1]'_{21}[\omega_1,\dots,\omega_\nu]_{13}$,

(viii) $\zeta'(A_4\times B_1) = [(\nu+\frac{1}{2})^\mu]_{21}+$
$+\sum(-1)^{\Sigma\lambda_r}[\nu+\frac{1}{2}-\lambda_\mu,\dots,\nu+\frac{1}{2}-\lambda_1]_{21}[\omega_1,\dots,\omega_\nu]_{14}$,

(ix) $\zeta(A_3\times B_3) = [(\nu+\frac{1}{2})^\mu]_{23}[(\frac{1}{2})^\nu]_{13}+$
$+\sum[\nu+\frac{1}{2}-\lambda_\mu,\dots,\nu+\frac{1}{2}-\lambda_1]_{23}[\frac{1}{2}+\omega_1,\dots,\frac{1}{2}+\omega_\nu]_{13}$,

(x) $\zeta(A_4\times B_3) = [(\nu+\frac{1}{2})^\mu]_{23}[(\frac{1}{2})^\nu]_{14}+$
$+\sum(-1)^{\Sigma\lambda_r}[\nu+\frac{1}{2}-\lambda_\mu,\dots,\nu+\frac{1}{2}-\lambda_1]_{23}[\frac{1}{2}+\omega_1,\dots,\frac{1}{2}+\omega_\nu]_{14}$,

(xi) $\zeta(A_4\times B_4) = [(\nu+\frac{1}{2})^\mu]_{24}[(\frac{1}{2})^\nu]_{14}+$
$+\sum[\nu+\frac{1}{2}-\lambda_\mu,\dots,\nu+\frac{1}{2}-\lambda_1]_{24}[\frac{1}{2}+\omega_1,\dots,\frac{1}{2}+\omega_\nu]_{14}$,

where $(\omega_1,\dots,\omega_\nu)$ *is the partition conjugate to* $(\lambda_1,\dots,\lambda_\mu)$.

The proofs of (i) and (ix) are given in (4) while, for the proof of (v), a slight adjustment in the argument of the proof of (4) Theorem III is needed. A summary only of the proofs of the remaining parts is given here.

(ii) From (3.1), we have

$$\zeta'(A_1\times B_1) = \prod_{i=1,j=1}^{\nu,\mu}(2\cos\phi_i-2\cos\theta_j)$$
$$= \frac{|(2\cos\phi_l)^{n-s}\,\vdots\,(2\cos\theta_\tau)^{n-s}|}{|(2\cos\phi_l)^{\mu-s_1}||(2\cos\theta_\tau)^{\mu+\nu-s_2}|},$$

where s_1 denotes the rows 1, 2,..., μ; s_2 denotes the rows $\mu+1,\dots,\mu+\nu$; s denotes the rows 1, 2,..., $n=\mu+\nu$; t denotes the columns 1, 2,..., μ; and τ the columns $\mu+1$, $\mu+2,\dots,\mu+\nu$.

Simplifying this determinant in the usual way, and taking the Laplace expansion of the numerator in terms of the first μ columns, we see that

$$\zeta'(A_1\times B_1) = \frac{|C_l^{n-s_1}||C_\tau^{n-s_2}|+\sum(-1)^p|C_l^{n-a_r}||C_\tau^{n-b_s}|}{|C_l^{\mu-s_1}||C_\tau^{n-s_2}|},$$

where p, a_r, b_s are defined as in (4) 330, and

$$C_l^r = 2\cos r\phi_l, \quad C_l^0 = 1; \qquad C_\tau^r = 2\cos r\theta_\tau, \quad C_\tau^0 = 1.$$

Thus, we prove

$$\zeta'(A_1\times B_1) = [\nu^\mu]_{21}+\sum(-1)^{\Sigma\lambda_r}[\nu-\lambda_\mu,\dots,\nu-\lambda_1]_{21}[\omega_1,\dots,\omega_\nu]_{11},$$

where $(\omega_1,\dots,\omega_\nu)$ is the partition conjugate to $(\lambda_1,\dots,\lambda_\mu)$.

(iii) From (3.12), we have

$$\zeta(A_1\times B_2) = \prod_{i=1,j=1}^{\mu-1,\nu}(2\cos\phi_i+2\cos\theta_j)\prod_{j=1}^{\nu}(2\sin\theta_j)$$

$$= \frac{|C_t^{n-s}\,|\,(-1)^{n-s}C_\tau^{n-s}|}{|C_t^{\mu-s_1}||(-1)^{n-s_2}C_\tau^{n-s_2}|}\prod_{j=1}^{\nu}(2\sin\theta_j),$$

where $n=\mu+\nu-1$. Multiply the numerator and denominator by $\prod_{i=1}^{\mu}(2\sin\phi_i)$; then we prove that

$$\zeta(A_1\times B_2) = \frac{|S_t^{n-s+1}\,|\,(-1)^{n-s}S_\tau^{n-s+1}|}{|S_t^{\mu-s_1+1}||(-1)^{n-s_2}C_\tau^{n-s_2}|}$$

$$= [(\nu-1)^\mu]_{22}[1^\nu]'_{11}+$$

$$+\sum[\nu-1-\lambda_\mu,\ldots,\nu-1-\lambda_1]_{22}[1+\omega_1,\ldots,1+\omega_\nu]'_{11}.$$

(iv) From (3.12), we have that

$$\zeta'(A_1\times B_2) = \prod_{i=1,j=1}^{\mu-1,\nu}(2\cos\phi_i-2\cos\theta_j)\prod_{j=1}^{\nu}(2\sin\theta_j).$$

Hence, in the same way, we prove that

$$\zeta'(A_1\times B_2) = [(\nu-1)^\mu]_{22}[1^\nu]'_{11}+$$

$$+\sum(-1)^{\Sigma\lambda_\nu}[\nu-1-\lambda_\mu,\ldots,\nu-1-\lambda_1]_{22}[1+\omega_1,\ldots,1+\omega_\nu]'_{11}.$$

(vi) In the usual way, we obtain that

$$\zeta'(A_3\times B_1) = \prod_{i=1,j=1}^{\mu,\nu}(2\cos\phi_i-2\cos\theta_j)\prod_{i=1}^{\mu}(2\sin\tfrac{1}{2}\phi_i)$$

$$= \frac{|C_t^{n-s}\,|\,C_\tau^{n-s}|}{|C_t^{\mu-s_1}||C_\tau^{n-s_2}|}\prod_{i=1}^{\mu}(2\sin\tfrac{1}{2}\phi_i).$$

Multiply denominator and numerator by $\prod_{j=1}^{\nu}(2\cos\frac{1}{2}\theta_j)$ and $\prod_{i=1}^{\mu}(2\cos\frac{1}{2}\phi_i)$ and simplify to give

$$\zeta'(A_3\times B_1)$$

$$= \frac{|C_t^{n-s+\frac{1}{2}}\,|\,C_\tau^{n-s+\frac{1}{2}}|}{|S_t^{\mu-s_1}||C_\tau^{n-s_2+\frac{1}{2}}|}$$

$$= [(\nu+\tfrac{1}{2})^\mu]'_{21}+\sum(-1)^{\Sigma\lambda_\nu}[\nu+\tfrac{1}{2}-\lambda_\mu,\ldots,\nu+\tfrac{1}{2}-\lambda_1]'_{21}[\tfrac{1}{2}+\omega_1,\ldots,\tfrac{1}{2}+\omega_\nu]_{14}.$$

In this case, we notice that the spin character of the direct product of the two matrices is given in terms of characters of different matrices.

(vii) In this case, we have

$$\zeta(A_4\times B_1) = \prod_{i=1,j=1}^{\mu,\nu}(2\cos\phi_i+2\cos\theta_j)\prod_{i=1}^{\mu}(2\sin\tfrac{1}{2}\phi_i)$$
$$= \frac{|C_t^{n-s+\frac{1}{2}} \vdots (-1)^{n-s}S_\tau^{n-s+\frac{1}{2}}|}{|S_t^{\mu-s_1}||(-1)^{n-s_2}S_\tau^{n-s_2+\frac{1}{2}}|},$$

where $S_t^r = 2\sin r\phi_t$ $(r\neq 0)$, $S_t^0 = \cot\tfrac{1}{2}\phi_t$.

Thus we prove that

$$\zeta(A_4\times B_1) = [(\nu+\tfrac{1}{2})^\mu]'_{21} + \sum[\nu+\tfrac{1}{2}-\lambda_\mu,\ldots,\nu+\tfrac{1}{2}-\lambda_1]'_{21}[\omega_1,\ldots,\omega_\nu]_{13}.$$

(viii) Similarly, we obtain

$$\zeta'(A_4\times B_1) = \prod_{i=1,j=1}^{\mu,\nu}(2\cos\phi_i-2\cos\theta_j)\prod_{i=1}^{\mu}(2\cos\tfrac{1}{2}\phi_i)$$
$$= \frac{|C_t^{n-s+\frac{1}{2}} \vdots C_\tau^{n-s+\frac{1}{2}}|}{|C_t^{\mu-s_1}||C_\tau^{n-s_2+\frac{1}{2}}|} = [(\nu+\tfrac{1}{2})^\mu]_{21} +$$
$$+\sum(-1)^{\Sigma\lambda_r}[\nu+\tfrac{1}{2}-\lambda_\mu,\ldots,\nu+\tfrac{1}{2}-\lambda_1]_{21}[\omega_1,\ldots,\omega_\nu]_{14}.$$

(x) In the usual way, we find that

$$\zeta(A_4\times B_3) = \prod_{i=1,j=1}^{\mu,\nu}(2\cos\phi_i-2\cos\theta_j)\prod_{i=1}^{\mu}(2\cos\tfrac{1}{2}\phi_i)\prod_{j=1}^{\nu}(2\sin\tfrac{1}{2}\theta_j)$$
$$= \frac{|S_t^{n-s+1} \vdots S_\tau^{n-s+1}|}{|S_t^{\mu-s_1+\frac{1}{2}}||C_\tau^{n-s_2+\frac{1}{2}}|} = [(\nu+\tfrac{1}{2})^\mu]_{23}[(\tfrac{1}{2})^\nu]_{14} +$$
$$+\sum(-1)^{\Sigma\lambda_r}[\nu+\tfrac{1}{2}-\lambda_\mu,\ldots,\nu+\tfrac{1}{2}-\lambda_1]_{23}[\tfrac{1}{2}+\omega_1,\ldots,\tfrac{1}{2}+\omega_\nu]_{24}.$$

(xi) Finally, we obtain that

$$\zeta(A_4\times B_4) = \prod_{i=1,j=1}^{\mu,\nu}(2\cos\phi_i+2\cos\theta_j)\prod_{i=1}^{\mu}(2\sin\tfrac{1}{2}\phi_i)\prod_{j=1}^{\nu}(2\sin\tfrac{1}{2}\theta_j)$$
$$= \frac{|S_t^{n-s+1} \vdots (-1)^{n-s}S_\tau^{n-s+1}|}{|C_t^{\mu-s_1+\frac{1}{2}}||(-1)^{n-s_2}C_\tau^{n-s_2+\frac{1}{2}}|} = [(\nu+\tfrac{1}{2})^\mu]_{24}[(\tfrac{1}{2})^\nu]_{14} +$$
$$+\sum[\nu+\tfrac{1}{2}-\lambda_\mu,\ldots,\nu+\tfrac{1}{2}-\lambda_1]_{24}[\tfrac{1}{2}+\omega_1,\ldots,\tfrac{1}{2}+\omega_\nu]_{14}.$$

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STRONG PROJECTIVE CONVERGENCE IN FUNCTION SPACES

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1. Introduction and preliminaries

ANALOGUES for function spaces of some of the results on sequence spaces given in (1) Chapter 10 have been obtained in (3), and, as stated there, those portions of the theories for sequence spaces have been considered there which deal with dual spaces, coordinate convergence, and projective convergence and limit. The object of this paper is to find analogues for function spaces of some of the other results on sequence spaces, as given in (1) Chapter 10, and so projective-boundedness, strong projective convergence, and distance-convergence are considered. Also, because of the importance of continuity in the study of functions, the concepts of strong projective continuity and distance-continuity are here introduced.

Definitions of the function spaces used in this paper, and of dual space, parametric convergence, projective convergence, and projective continuity are to be found in (3).

A linear space S is said to be *normed* if to each element e of S there corresponds a non-negative number $\|e\|$ which is called the *norm* of e, and is such that

(i) $\|e\| = 0$ if and only if $e = 0$,

(ii) $\|ce\| = |c| . \|e\|$ for every scalar c,

(iii) $\|e+f\| \leqslant \|e\|+\|f\|$.

For example, in $\sigma_p(f)$ $(1 \leqslant p < \infty)$, the norm is given by

$$\|f\|_p = \left(\int_0^\infty |f(x)|^p \, dx \right)^{1/p},$$

and in $\sigma_\infty(f)$ it is given by

$$\|f\|_\infty = \operatorname*{ess\,sup}_{x \in [0,\infty)} |f(x)|.$$

If, for a fixed $t > 0$,

$$f_t(x) = \begin{cases} \psi(x) & (0 \leqslant x \leqslant t), \\ 0 & (x > t), \end{cases}$$

then $f_t(x)$ is called a *section* of $\psi(x)$.

Quart. J. Math. Oxford (2), 12 (1961), 177–84.

If $\alpha(f)$ is a function space, sections of functions in $\alpha(f)$ may or may not belong to it; for example, sections of functions in $\sigma_\infty(f)$, $\sigma_p(f)$, where $1 \leqslant p < \infty$, $\Gamma(f)$, $\phi(f)$, and $V(f)$ belong respectively to these spaces, whereas sections of functions in $P(f)$ do not belong to $P(f)$, and also sections of constant functions do not belong to the space of constant functions.

2. Projective-bounded sets

If $\alpha^*(f) \geqslant \beta(f)$, and if

$$\left| \int_0^\infty f(x)g(x)\,dx \right| \leqslant \kappa(g)$$

for every f in a set X in $\alpha(f)$ and every g in $\beta(f)$, where $\kappa(g)$ is a positive constant depending on g, then we say that the set X is *projective-bounded* (p-bd) *relative to* $\beta(f)$, or $\alpha(f)\beta(f)$-bd. When $\beta(f) = \alpha^*(f)$, we say that X is p-bd *in* $\alpha(f)$, or $\alpha(f)$-bd [cf. (**1**) 293].

If $\alpha^*(f) \geqslant \beta(f)$, and we take a set X in $\alpha(f)$ to be the family $f_t(x)$ with $t \in [0, \infty)$, then we say that $f_t(x)$ is '$\alpha(f)\beta(f)$-bd' if

$$\left| \int_0^\infty f_t(x)g(x)\,dx \right| \leqslant \kappa(g)$$

for all t in $[0, \infty)$ and every g in $\beta(f)$.

It is obvious from the definition of $\alpha(f)\beta(f)$-convergence given in (**3**), and that of $\alpha(f)\beta(f)$-boundedness given above, that $\alpha(f)\beta(f)$-convergence does not necessarily imply $\alpha(f)\beta(f)$-boundedness. This makes a fundamental difference from the case of sequence spaces, where $\alpha\beta$-convergence always implies $\alpha\beta$-boundedness [see (**1**) 293].

(2, I). *A set X is* $\sigma_p(f)$-bd *if and only if*

$$\|f\|_p \leqslant M$$

for every f in X, where $1 \leqslant p \leqslant \infty$.

Let $f \in X$ in $\sigma_p(f)$, and let g be any function in $\sigma_p^*(f) = \sigma_q(f)$, where $1/p + 1/q = 1$. That the condition is sufficient follows at once from Hölder's inequality

$$\left| \int_0^\infty f(x)g(x)\,dx \right| \leqslant \|f\|_p \cdot \|g\|_q.$$

We now prove the necessity of the condition.

If X is $\sigma_p(f)$-bd, then

$$|F_f(g)| = \left| \int_0^\infty f(x)g(x)\,dx \right| \leqslant \kappa(g) \tag{2.1}$$

for every f in X and every g in $\sigma_q(f)$.

$F_f(g)$ is a scalar for every f in $\sigma_p(f)$ and every g in $\sigma_q(f)$; therefore $\|F_f(g)\| = |F_f(g)|$, and so, by (2.1),

$$\|F_f(g)\| \leqslant \kappa(g) \tag{2.2}$$

for each g in $\sigma_q(f)$, as f runs through the set X.

Hence it follows from (2.2), and by the Banach–Steinhaus theorem on uniform boundedness [see **(1)** 319–20, **(4)**, or **(5)** 135], that

$$\|F_f(g)\| \leqslant M \cdot \|g\|_q \tag{2.3}$$

for every f in X and every g in $\sigma_q(f)$.

Now, for each f in $\sigma_p(f)$,

$$F_f(g) = \int_0^\infty f(x)g(x)\,dx$$

is a bounded linear functional on $\sigma_q(f)$.

Hence, from (2.3), by the definition of the norm of a bounded linear functional [see **(2)** 350 or **(5)** 138], we have

$$\|F_f\| \leqslant M \tag{2.4}$$

for every f in X; and also, for each f in X, we have

$$\|F_f\| = \sup|F_f(g)|$$

for all g in $\sigma_q(f)$ with $\|g\|_q \leqslant 1$; i.e.

$$\|F_f\| = \sup\left| \int_0^\infty f(x)g(x)\,dx \right| \tag{2.5}$$

for all g in $\sigma_q(f)$ with $\|g\|_q \leqslant 1$.

But, by **(5)** [Theorem 2, 71–72],

$$\|f\|_p = \sup\left| \int_0^\infty f(x)g(x)\,dx \right| \tag{2.6}$$

for all g in $\sigma_q(f)$ with $\|g\|_q \leqslant 1$.

Hence, by (2.5) and (2.6), we have

$$\|F_f\| = \|f\|_p. \tag{2.7}$$

Therefore, by (2.4) and (2.7),

$$\|f\|_p \leqslant M$$

for every f in X; and thus the condition is necessary.

COROLLARY 1. *If* $1 \leqslant p < \infty$, *a set* X *is* $\sigma_p(f)$-bd *if and only if*

$$\int_0^\infty |f(x)|^p \, dx \leqslant M^p$$

for every f *in* X [cf. (1) 298 (10.4, IV), and (299) (10.4, V)].

COROLLARY 2. *A set* X *is* p-bd *in* $\sigma_\infty(f)$ *if and only if*

$$|f(x)| \leqslant M$$

for almost all $x \geqslant 0$ *and every* f *in* X [cf. (1) 298 (10.4, III)].

3. Strong projective convergence, and strong projective continuity

If $\alpha^*(f) \geqslant \beta(f)$, and $f_t(x)$ in $\alpha(f)$ satisfies the condition that to every $\epsilon > 0$ and every p-bd set U in $\beta(f)$ corresponds a positive number $T(\epsilon, U)$ such that

$$\left| \int_0^\infty g(x)\{f_t(x) - f_{t'}(x)\} \, dx \right| \leqslant \epsilon \tag{3.1}$$

for every g in U and all $t, t' \geqslant T(\epsilon, U)$, then $f_t(x)$ is said to be *strongly projective convergent* (**p**-cgt) *relative to* $\beta(f)$, or $\boldsymbol{\alpha(f)\beta(f)}$-cgt. When $\beta(f) = \alpha^*(f)$, we say that $f_t(x)$ is **p**-cgt *in* $\alpha(f)$, or $\boldsymbol{\alpha(f)}$-cgt [cf. (1) 302].

If $\alpha^*(f) \geqslant \beta(f)$, and $f_t(x)$ in $\alpha(f)$ satisfies the condition that to every $\epsilon > 0$ and every p-bd set U in $\beta(f)$ corresponds a positive number $\delta(\epsilon, U)$ such that (3.1) holds for every g in U and all non-negative t and t' such that $|t - t'| \leqslant \delta(\epsilon, U)$, then $f_t(x)$ is said to be *strongly projective continuous* (**p**-*continuous*) *relative to* $\beta(f)$, or $\boldsymbol{\alpha(f)\beta(f)}$-*continuous*. When $\beta(f) = \alpha^*(f)$, we say that $f_t(x)$ is **p**-*continuous in* $\alpha(f)$, or $\boldsymbol{\alpha(f)}$-*continuous*.

By taking U to consist of one function only, we see from the definitions that $\boldsymbol{\alpha(f)\beta(f)}$-convergence implies $\alpha(f)\beta(f)$-convergence, and that $\boldsymbol{\alpha(f)\beta(f)}$-continuity implies $\alpha(f)\beta(f)$-continuity.

(3, I). *Every parametric convergent family in* $\sigma_\infty(f)$ *is* **p**-cgt *in* $\sigma_\infty(f)$.

We have $\sigma_\infty^*(f) = \sigma_1(f)$. Let U be a p-bd set in $\sigma_1(f)$; then, by (2, I) Corollary 1,

$$\int_0^\infty |g(x)| \, dx \leqslant M \tag{3.2}$$

for every g in U.

Let $f_t(x)$ belong to $\sigma_\infty(f)$ and be parametric convergent (t-cgt); then corresponding to every $\epsilon > 0$ we can choose a positive number $T(\epsilon)$, independent of x, such that, for almost all $x \geqslant 0$,

$$|f_t(x) - f_{t'}(x)| \leqslant \frac{\epsilon}{M} \tag{3.3}$$

for all $t, t' \geqslant T(\epsilon)$.

Hence, by (3.2) and (3.3),

$$\left|\int_0^\infty g(x)\{f_t(x)-f_{t'}(x)\}\,dx\right| \leqslant \frac{\epsilon}{M}\int_0^\infty |g(x)|\,dx \leqslant \epsilon$$

for every g in U and all t, $t' \geqslant T(\epsilon)$, and for every $\epsilon > 0$.

Therefore $f_t(x)$ is **p**-cgt in $\sigma_\infty(f)$.

We say that '$f_t(x) \in \mathscr{C}_1(f)_t$' if to every $\epsilon > 0$ there corresponds a positive number $\delta(\epsilon)$, independent of t and x, such that, for almost all $x \geqslant 0$,

$$|f_t(x)-f_{t'}(x)| \leqslant \epsilon$$

for all non-negative t and t' satisfying $|t-t'| \leqslant \delta(\epsilon)$.

(3, II). *Every family $f_t(x)$ in $\sigma_\infty(f)$, such that $f_t(x) \in \mathscr{C}_1(f)_t$, is* $\boldsymbol{\sigma}_\infty(\mathbf{f})$*-continuous.*

The proof is similar to that of (3, I).

(3, III). *For every parametric convergent family in $\sigma_1(f)$, p-convergence and* **p***-convergence coincide* [cf. (**1**) 304 (10.5, II)].

We have $\sigma_1^*(f) = \sigma_\infty(f)$. If U is a p-bd set in $\sigma_\infty(f)$, then, by (2, I) Corollary 2,

$$|g(x)| \leqslant M \tag{3.4}$$

for every g in U and almost all $x \geqslant 0$.

Let $f_t(x)$ belong to $\sigma_1(f)$ and be t-cgt, and suppose that it is $\sigma_1(f)$-cgt. Then, since $\sigma_\infty(f)$ is normal† and $f_t(x)$ is t-cgt, we can, by [(**3**) (4, V)], taking $g(x) = 1$ for all $x \geqslant 0$, for every $\epsilon > 0$, determine a positive number $T(\epsilon)$ such that

$$\int_0^\infty |f_t(x)-f_{t'}(x)|\,dx \leqslant \frac{\epsilon}{M} \tag{3.5}$$

for all t, $t' \geqslant T$.

Hence, by (3.4) and (3.5),

$$\left|\int_0^\infty g(x)\{f_t(x)-f_{t'}(x)\}\,dx\right| \leqslant M\int_0^\infty |f_t(x)-f_{t'}(x)|\,dx \leqslant \epsilon$$

for every g in U and all t, $t' \geqslant T$.

Therefore $f_t(x)$ is $\boldsymbol{\sigma}_1(\mathbf{f})$-convergent.

Also **p**-convergence implies p-convergence.

Hence the result follows.

(3, IV). *For every family $f_t(x)$ in $\sigma_1(f)$ such that $f_t(x) \in \mathscr{C}_1(f)_t$, p-continuity and* **p***-continuity coincide.*

The proof is similar to that of (3, III).

† We refer to (**3**) for the definition of a normal function space.

(3, V). *When $\beta(f)$ is normal and is such that sections of functions in it belong to it, then the necessary and sufficient condition that $f_t(x)$ in $\alpha(f)$ should be* $\boldsymbol{\alpha(f)\beta(f)}$-cgt *is that to every $\epsilon > 0$ and every* p-bd *set U in $\beta(f)$, there corresponds a positive number $T(\epsilon, U)$ such that*

$$\int_0^\infty |g(x)\{f_t(x)-f_{t'}(x)\}|\,dx \leqslant \epsilon$$

for every g in U and all $t, t' \geqslant T$ [cf. **(1) 303** (10.5, I)].

If follows from the definition of **p**-convergence that the condition is sufficient.

To prove that it is necessary, we construct the sections

$$h_r(x) = \begin{cases} g(x) & (0 \leqslant x \leqslant r), \\ 0 & (x > r), \end{cases}$$

of every g in U, and consider the set V which consists of all functions $\theta_r(x)$ such that $|\theta_r(x)| = |h_r(x)|$ for all $x \geqslant 0$. By hypothesis, $\beta(f)$ is such that sections of functions in it belong to it; hence $h_r(x) \in \beta(f)$; also $\beta(f)$ is normal, so that $\theta_r(x) \in \beta(f)$.

For all $\omega(x)$ in $\beta^*(f)$,

$$\left|\int_0^\infty \theta_r(x)\omega(x)\,dx\right| \leqslant \int_0^\infty |\theta_r(x)\omega(x)|\,dx = \int_0^\infty |h_r(x)\omega(x)|\,dx$$

$$= \int_0^r |g(x)\omega(x)|\,dx \leqslant \int_0^\infty |g(x)\omega(x)|\,dx$$

for every $r > 0$; hence the set V is p-bd in $\beta(f)$. Consequently, if $f_t(x)$ is $\boldsymbol{\alpha(f)\beta(f)}$-cgt,

$$\left|\int_0^\infty \theta_r(x)\{f_t(x)-f_{t'}(x)\}\,dx\right| \leqslant \epsilon$$

for all $t, t' \geqslant T(\epsilon, U)$ for every $\epsilon > 0$.

Given any two fixed numbers $t, t' \geqslant T$, we choose the signs of $\theta_r(x)$ so that $\theta_r(x)\{f_t(x)-f_{t'}(x)\}$ is non-negative. Then we have, for all $t, t' \geqslant T(\epsilon, U)$,

$$\int_0^r |g(x)\{f_t(x)-f_{t'}(x)\}|\,dx \leqslant \epsilon$$

for every $r > 0$ and every g in U; the result thus follows.

In (3, V), unlike the case of sequence spaces [see **(1) 303** (10.5, I)], we have to make the additional hypothesis that $\beta(f)$ is normal, and is such that sections of functions in it belong to it, because sections of a sequence always belong to the sequence space ϕ, whereas the sections of a

function do not necessarily belong to the function space $\phi(f)$, since a section of a function is not necessarily essentially bounded; but functions in $\phi(f)$ are, by definition [see (3)], essentially bounded.

4. Distance convergence, and distance continuity

If with every two functions $f(x)$ and $g(x)$ in a space $\alpha(f)$, we can associate a number $d(f,g)$ such that Hausdorff's postulates

(i) $d(f,f)=0, \quad d(f,g)>0 \quad$ when $f \neq g$,

(ii) $d(f,g)=d(g,f)$,

(iii) $d(f,h) \leqslant d(f,g)+d(g,h), \quad$ where $h \in \alpha(f)$,

are satisfied, then $d(f,g)$ may be called the *distance between f and g* in $\alpha(f)$, and $\alpha(f)$ is then a *metric space*.

In $\sigma_p(f)$ $(1 \leqslant p < \infty)$ distance is defined by

$$d(f,g)=\left(\int_0^\infty |f(x)-g(x)|^p\,dx\right)^{1/p} \equiv \|f-g\|_p; \tag{4.1}$$

(i), (ii), and (iii) are then satisfied.

$f_t(x)$ in a metric space is said to be *distance-convergent* (d-cgt) when to every $\epsilon > 0$ corresponds a positive number $T(\epsilon)$ such that

$$d(f_t,f_{t'}) \leqslant \epsilon \tag{4.2}$$

for all $t, t' \geqslant T(\epsilon)$.

$f_t(x)$ in a metric space is said to be *distance-continuous* (*d-continuous*) when given any $\epsilon > 0$, there exists a positive number $\delta(\epsilon)$, independent of t, such that (4.2) holds for all non-negative t and t' satisfying

$$|t-t'| \leqslant \delta(\epsilon).$$

(4, I). *For every parametric convergent family in $\sigma_1(f)$, d-convergence, p-convergence, and* **p**-*convergence coincide* [cf. **(1) 316 (10.8, II)**].

By (3, III), for every parametric convergent family in $\sigma_1(f)$, p-convergence and p-convergence coincide.

Let $f_t(x)$ in $\sigma_1(f)$ be t-cgt.

We have $\sigma_1^*(f)=\sigma_\infty(f)$. Suppose that $f_t(x)$ is p-cgt in $\sigma_1(f)$. Then, by (3) [(4, V)], taking $g(x)=1$ for all $x \geqslant 0$, corresponding to every $\epsilon > 0$ we can determine a positive number $T(\epsilon)$ such that

$$\int_0^\infty |f_t(x)-f_{t'}(x)|\,dx \leqslant \epsilon \tag{4.3}$$

for all $t, t' \geqslant T$; and, by (4.1) and (4.2), this is the definition of d-convergence of $f_t(x)$ in $\sigma_1(f)$; hence $f_t(x)$ is d-cgt.

Now suppose that $f_t(x)$ is d-cgt; then (4.3) is satisfied. Therefore for any g in $\sigma_\infty(f)$, we have, by (4.3),

$$\left|\int_0^\infty g(x)\{f_t(x)-f_{t'}(x)\}\,dx\right| \leqslant M(g)\int_0^\infty |f_t(x)-f_{t'}(x)|\,dx$$
$$\leqslant \epsilon M(g)$$

for all $t, t' \geqslant T(\epsilon)$ for every $\epsilon > 0$, and so $f_t(x)$ is p-cgt in $\sigma_1(f)$.

Thus the theorem is established.

(4, II). *For every family $f_t(x)$ in $\sigma_1(f)$, such that $f_t(x) \in \mathscr{C}_1(f)_t$, d-continuity, p-continuity, and* **p**-*continuity coincide.*

The proof is similar to that of (4, I).

I wish to express my thanks to Dr. R. G. Cooke for some useful suggestions.

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ON TRANSFORMATIONS OF FUNCTIONS AND FUNCTION SPACES

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1. Introduction and preliminaries

THE object of this paper is to consider some transformations of function spaces into function spaces, and also to study the nature of transforms of functions of the type $g(x, r)$, where r is a (non-negative continuous variable) parameter. Sequence to sequence, and sequence space to sequence space, transformations are carried out by means of an infinite matrix $(a_{n,k})$ [see (1) Chapters 4 and 10 and (2) Chapter 6]. A kernel $F(x, t)$ is employed to transform a function to a function, and a function space into a function space.

$F(x, t)$ is said to *transform* $g(x)$ if the integral

$$\int_0^\infty F(x, t)g(x)\,dx \tag{1.1}$$

is absolutely convergent for all t in $[0, \infty)$, and then

$$\psi_g(t) = \int_0^\infty F(x, t)g(x)\,dx \tag{1.2}$$

is called the *F-transform* of $g(x)$.

$F(x, t)$ is said to *transform* a function space $\alpha(f)$ if the integral (1.1) is absolutely convergent for all t in $[0, \infty)$ and for every $g(x)$ in $\alpha(f)$, and then the functions $\psi_g(t)$, given by (1.2), where $g(x) \in \alpha(f)$, are called the *F-transforms of the functions in* $\alpha(f)$; and, if these F-transforms $\psi_g(t)$ belong to a function space $\beta(f)$, then we say that the *function space* $\alpha(f)$ *has been transformed into the function space* $\beta(f)$, or that the *F-transform of* $\alpha(f)$ *is in* $\beta(f)$.

With t as a (continuous variable) parameter, if $F(x, t)$, regarded as a function of x, belongs to $\alpha(f)$ for all t in $[0, \infty)$, then we write $F(x, t) \in \alpha(f)_x$; if further $\alpha(f) \leqslant \beta(f)$, then we write

$$F(x, t) \in \alpha(f)_x \leqslant \beta(f).$$

It follows from the definition of dual space, given in (3), and that of the F-transform of $\alpha(f)$, given above, that $F(x, t)$ transforms $\alpha(f)$ if and only if $F(x, t) \in \alpha^*(f)_x$.

Quart. J. Math. Oxford (2), 12 (1961), 185–95.

We refer to (3) for the definitions of the function spaces used in this paper but not defined in it.

Let $f(x,t) \in \sigma_p(f)_x$, where $1 \leqslant p \leqslant \infty$. If, corresponding to every $\epsilon > 0$, there exists a positive number $T(\epsilon)$ such that, writing

$$f(x,t) \equiv f_t(x),$$

we have† $$\|f_t - f_{t'}\|_p \leqslant \epsilon \tag{1.3}$$

for all $t, t' \geqslant T(\epsilon)$, then we say that $f_t(x)$ *converges in mean* (or *converges in norm*) *with exponent* p. If, corresponding to every $\epsilon > 0$, there exists a positive number $\delta(\epsilon)$ such that (1.3) holds for all non-negative t and t' satisfying $|t-t'| \leqslant \delta(\epsilon)$, we say that $f_t(x)$ is *continuous in mean* (or *continuous in norm*) *with exponent* p. If

$$\|f_t\|_p \leqslant M \tag{1.4}$$

for all $t \geqslant 0$, we say that $f_t(x)$ is *bounded in mean* (or *bounded in norm*) *with exponent* p.

In the case $1 \leqslant p < \infty$, (1.3) and (1.4) are respectively

$$\int_0^\infty |f_t(x) - f_{t'}(x)|^p \, dx \leqslant \epsilon^p \tag{1.3 a}$$

and

$$\int_0^\infty |f_t(x)|^p \, dx \leqslant M^p. \tag{1.4 a}$$

2. We now establish a few theorems on transformations of function spaces into function spaces.

(2, I). *If* $F(x,t) \geqslant 0$ *for almost all* $x \geqslant 0$ *and almost all* $t \geqslant 0$,

$$\alpha(f) \leqslant \sigma_\infty(f),$$

and $\alpha(f)$ *contains all functions which are constant throughout* $[0, \infty)$, *then* $F(x,t)$ *transforms* $\alpha(f)$ *into a normal space* $\beta(f)$ *if and only if*

(i) $F(x,t) \in \sigma_1(f)_x$,

(ii) $\int_0^\infty F(x,t)\,dx \in \beta(f)$.

The proof is similar to that of [(4) (3, I)].

We refer to (3) for the definition of a normal function space. For example, the spaces $\sigma_p(f)$, where $1 \leqslant p \leqslant \infty$, $\phi(f)$, $E_r(f)$, and $F_r(f)$ are normal, so that, in (2, I), $\beta(f)$ could be any one of these spaces.

'$F(x,t) \in \mathscr{M}(f)_t$' has the same meaning as '$F_t(x) \in \mathscr{M}(f)_t$', and for this we refer to (3).

† We refer to (5) for the meaning of $\|f\|_p$.

Let $\mathscr{C}_0(f)$ denote the space of all functions uniformly continuous in $[0,\infty)$.

(2, II). *If $F(x,t)\in\mathscr{M}(f)_t$, then $F(x,t)$ transforms $\sigma_\infty(f)$ into $\mathscr{C}_0(f)$ if and only if $F(x,t)$ belongs to $\sigma_1(f)_x$ and is continuous in mean with exponent* 1.

Suppose that $F(x,t)\in\sigma_1(f)_x$; then $F(x,t)$ transforms $\sigma_\infty(f)$ since $\sigma^*_\infty(f)=\sigma_1(f)$.

Also suppose that $F(x,t)$ is continuous in mean with exponent 1; then to every $\epsilon>0$ there corresponds $\delta(\epsilon)>0$ such that, by (1.3 a),

$$\int_0^\infty |F(x,t)-F(x,t')|\,dx\leqslant\epsilon \tag{2.1}$$

for all non-negative t and t' satisfying $|t-t'|\leqslant\delta(\epsilon)$.

Let $g(x)\in\sigma_\infty(f)$, and let $\psi_g(t)$ be given by (1.2); then, by (2.1), given any $\epsilon>0$, there exists $\delta(\epsilon)>0$ such that, for all non-negative t and t' satisfying $|t-t'|\leqslant\delta(\epsilon)$,

$$\begin{aligned}|\psi_g(t)-\psi_g(t')| &= \left|\int_0^\infty \{F(x,t)-F(x,t')\}g(x)\,dx\right| \\ &\leqslant M(g)\int_0^\infty |F(x,t)-F(x,t')|\,dx\leqslant\epsilon M(g).\end{aligned}$$

Hence $\psi_g(t)\in\mathscr{C}_0(f)$ for every $g(x)$ in $\sigma_\infty(f)$; and so the conditions are sufficient.

Conversely, if $F(x,t)$ transforms $\sigma_\infty(f)$, then $F(x,t)$ must belong to $\sigma^*_\infty(f)_x=\sigma_1(f)_x$; and, if the F-transform $\psi_g(t)$ belongs to $\mathscr{C}_0(f)$ for every $g(x)$ in $\sigma_\infty(f)$, then to every $\epsilon>0$ and to every $g(x)$ in $\sigma_\infty(f)$, there corresponds a positive number $\delta(\epsilon,g)$ such that

$$\left|\int_0^\infty \{F(x,t)-F(x,t')\}g(x)\,dx\right|\leqslant\epsilon \tag{2.2}$$

for all non-negative t and t' such that $|t-t'|\leqslant\delta$.

Take $g(x)=1$ for all $x\geqslant 0$; then, since, by hypothesis, $F(x,t)\in\mathscr{M}(f)_t$, we have, by (2.2),

$$\int_0^\infty |F(x,t)-F(x,t')|\,dx\leqslant\epsilon$$

for all non-negative t and t' such that $|t-t'|\leqslant\delta(\epsilon)$, for every $\epsilon>0$; i.e. $F(x,t)$ is continuous in mean with exponent 1. Thus the conditions are necessary.

Let $\Gamma_0(f)$ denote the space of all functions $f(x)$ which are such that

(i) $f(x)$ is finite everywhere in $[0,\infty)$ and (ii) $f(x)$ tends to a finite limit as $x \to \infty$.

(2, III). *If $F(x,t) \in \mathscr{M}(f)_t$, then the F-transform of $\sigma_\infty(f)$ is in $\Gamma_0(f)$ if and only if $F(x,t)$ belongs to $\sigma_1(f)_x$ and is convergent in mean with exponent* 1.

The proof is similar to that of (2, II).

Let $\sigma'_\infty(f)$ denote the space of functions bounded in $[0,\infty)$, to distinguish this from $\sigma_\infty(f)$.

(2, IV). *If $1 \leqslant p \leqslant \infty$, a kernel $F(x,t)$ transforms $\sigma_p(f)$ into $\sigma'_\infty(f)$ if and only if $F(x,t)$ belongs to $\sigma_q(f)_x$ and is bounded in mean with exponent q, where $1/p+1/q = 1$.*

If $F(x,t) \in \sigma_q(f)_x$, then $F(x,t)$ transforms $\sigma_p(f)$ since $\sigma_p^*(f) = \sigma_q(f)$.

Let $g(x)$ be any function in $\sigma_p(f)$, and let $\psi_g(t)$ be given by (1.2). Let us write $F(x,t) \equiv F_t(x)$.

If $F(x,t)$ is bounded in mean with exponent q, then, by (1.4),

$$\|F_t\|_q \leqslant M \tag{2.3}$$

for all $t \geqslant 0$. Therefore, by Hölder's inequality and (2.3),

$$|\psi_g(t)| = \left| \int_0^\infty F(x,t)g(x)\,dx \right| \leqslant \|F_t\|_q \cdot \|g\|_p \leqslant M \cdot \|g\|_p$$

for all $t \geqslant 0$. Hence $\psi_g(t) \in \sigma'_\infty(f)$ for every $g(x)$ in $\sigma_p(f)$.

Thus the conditions are sufficient.

Conversely, if $F(x,t)$ transforms $\sigma_p(f)$, then $F(x,t)$ must belong to $\sigma_p^*(f)_x = \sigma_q(f)_x$; and, if the F-transform $\psi_g(t)$ belongs to $\sigma'_\infty(f)$ for every $g(x)$ in $\sigma_p(f)$, then

$$|\psi_g(t)| = \left| \int_0^\infty F(x,t)g(x)\,dx \right| \leqslant K(g) \tag{2.4}$$

for all $t \geqslant 0$ and every $g(x)$ in $\sigma_p(f)$, where $K(g)$ is a positive constant independent of t, but depending on g.

Now let us write

$$\phi_t(g) \equiv \psi_g(t) = \int_0^\infty F(x,t)g(x)\,dx$$

for every t in $[0,\infty)$ and every g in $\sigma_p(f)$. Then, by (2.4), we have

$$|\phi_t(g)| \leqslant K(g) \tag{2.5}$$

for all t in $[0,\infty)$ and every g in $\sigma_p(f)$. Now $\phi_t(g)$ is a scalar for each t in $[0,\infty)$ and every g in $\sigma_p(f)$; therefore $\|\phi_t(g)\| = |\phi_t(g)|$, and so, by (2.5),

$$\|\phi_t(g)\| \leqslant K(g)$$

for each g in $\sigma_p(f)$ as t runs through $[0,\infty)$.

Hence, by the Banach–Steinhaus theorem on uniform boundedness [see (1) 319–20, or (6) 135],

$$\|\phi_t(g)\| \leqslant K' \cdot \|g\|_p \tag{2.6}$$

for all t in $[0,\infty)$ and all g in $\sigma_p(f)$.

For each t in $[0,\infty)$,

$$\phi_t(g) = \int_0^\infty F_t(x)g(x)\,dx$$

is a bounded linear functional on $\sigma_p(f)$. Therefore, it follows, by (2.6), from the definition of the norm of a bounded linear functional [see (2) 350 or (6) 138] that

$$\|\phi_t\| \leqslant K' \tag{2.7}$$

for all t in $[0,\infty)$.

Also, by this same definition, we have, for each fixed t in $[0,\infty)$, $\|\phi_t\| = \sup|\phi_t(g)|$ for all g in $\sigma_p(f)$ with $\|g\|_p \leqslant 1$; i.e.

$$\|\phi_t\| = \sup\left|\int_0^\infty F_t(x)g(x)\,dx\right| \tag{2.8}$$

for all g in $\sigma_p(f)$ with $\|g\|_p \leqslant 1$.

But, by [(6) Theorem 2, 71–72], for each fixed t in $[0,\infty)$,

$$\|F_t\|_q = \sup\left|\int_0^\infty F_t(x)g(x)\,dx\right| \tag{2.9}$$

for all g in $\sigma_p(f)$ with $\|g\|_p \leqslant 1$. Hence, by (2.8) and (2.9), we have for each fixed t in $[0,\infty)$,

$$\|F_t\|_q = \|\phi_t\|. \tag{2.10}$$

Therefore, by (2.7) and (2.10), we have

$$\|F_t\|_q \leqslant K'$$

for all t in $[0,\infty)$; i.e. $F(x,t)$ is bounded in mean with exponent q. Thus the conditions are necessary.

In particular, we have

COROLLARY 1. *$F(x,t)$ transforms $\sigma_\infty(f)$ into $\sigma'_\infty(f)$ if and only if $F(x,t) \in \sigma_1(f)_x$ and*

$$\int_0^\infty |F(x,t)|\,dx \leqslant M \quad \textit{for all } t \textit{ in } [0,\infty).$$

COROLLARY 2. *$F(x,t)$ transforms $\sigma_1(f)$ into $\sigma'_\infty(f)$ if and only if $F(x,t) \in \sigma_\infty(f)_x$ and*

$$\operatorname*{ess\,sup}_{x\in[0,\infty)} |F(x,t)| \leqslant M$$

for all t in $[0,\infty)$.

3. In this section we study the nature of the transforms by a kernel $F(x,t)$ of functions of the type $g(x,r)$, where r is a (non-negative con-

tinuous variable) parameter, i.e. of families of the type $g(x,r) \equiv g_r(x)$, where r runs through the set $[0,\infty)$. We give below some notations and definitions which we shall need in connexion with this study.

In the notations given in § 1, $f(x,t) \equiv f_t(x)$ will be called $\alpha(f)_x\beta(f)$-cgt if $f_t(x)$ is $\alpha(f)\beta(f)$-cgt in the notations of (**3**); in the same way, we shall say that $f(x,t)$ is '$\alpha(f)_x\beta(f)$-continuous', '$\alpha(f)_x\beta(f)$-bd', '$\boldsymbol{\alpha(f)_x\beta(f)}$-cgt', or '$\boldsymbol{\alpha(f)_x\beta(f)}$-continuous' according as $f_t(x) \equiv f(x,t)$ is $\alpha(f)\beta(f)$-continuous, $\alpha(f)\beta(f)$-bd, $\boldsymbol{\alpha(f)\beta(f)}$-cgt, or $\boldsymbol{\alpha(f)\beta(f)}$-continuous, in the notations of (**3**) and (**5**).

We see in (**5**) that in the case of function spaces, unlike the case of sequence spaces [(**1**) 293], projective convergence does not necessarily imply projective boundedness. If $f(x,t)$ is $\alpha(f)_x\beta(f)$-cgt and also $\alpha(f)_x\beta(f)$-bd, then we say that $f(x,t)$ is '$\alpha(f)_x\beta(f)$-(bd cgt)'.

If $\alpha^*(f) \geqslant \beta(f)$, $F(x,t) \in \alpha(f)_x$, $g(x,r) \in \beta(f)_x$,

$$\psi_g(t,r) = \int_0^\infty F(x,t)g(x,r)\,dx, \tag{3.1}$$

and to every $\epsilon > 0$ and to every $g(x,r) \in \beta(f)_x$ there corresponds a positive number $T(\epsilon,g)$, independent of r, such that, for almost all $r \geqslant 0$,

$$|\psi_g(t,r)-\psi_g(t',r)| = \left|\int_0^\infty \{F(x,t)-F(x,t')\}g(x,r)\,dx\right| \leqslant \epsilon \tag{3.2}$$

for all $t, t' \geqslant T(\epsilon,g)$, then we say that $F(x,t)$ *in* $\alpha(f)_x$ is *smoothly projective convergent* {p-cgt(s)} *relative to* $\beta(f)$, or $\alpha(f)_x\beta(f)$-cgt(s). When $\beta(f) = \alpha^*(f)$, we say that $F(x,t)$ is '$\alpha(f)_x$-cgt(s)'.

If (3.2) is true for all non-negative t and t' such that $|t-t'| \leqslant \delta(\epsilon,g)$ for every $\epsilon > 0$ and every $g(x,r)$ in $\beta(f)_x$, where δ is independent of r and t, then $F(x,t)$ *in* $\alpha(f)_x$ is said to be *smoothly projective continuous* {*p-continuous(s)*} *relative to* $\beta(f)$, or $\alpha(f)_x\beta(f)$-*continuous(s)*. When $\beta(f) = \alpha^*(f)$, $F(x,t)$ is said to be $\alpha(f)_x$-*continuous(s)*.

$\psi_g(t,r)$, given by (3.1), is referred to as the 'F-transform of $g(x,r)$'.

We now prove some theorems on transforms.

(3, I). *If* $F(x,t)$ *is* $\alpha(f)_x\beta(f)$-cgt(s), *then the* F-*transform of every* $\beta(f)_x\alpha(f)$-(bd cgt) *family* $g(x,r)$ *is* $\boldsymbol{\Gamma(f)_r}$-cgt.

Let $\psi_g(t,r)$ be given by (3.1).

By hypothesis, $F(x,t)$ is $\alpha(f)_x\beta(f)$-cgt(s), so to every $\epsilon > 0$ and to every $g(x,r)$ in $\beta(f)_x$ there corresponds a positive number $T(\epsilon,g)$ such that (3.2) holds for almost all $r \geqslant 0$ and all $t, t' \geqslant T(\epsilon,g)$.

Since $g(x,r)$ is $\beta(f)_x\alpha(f)$-bd, we have, for all t in $[0,\infty)$,

$$|\psi_g(t,r)| \leqslant M(t) \quad \text{for all } r \text{ in } [0,\infty).$$

Also, $g(x,r)$ is $\beta(f)_x\alpha(f)$-cgt; therefore, for every t in $[0,\infty)$, $\psi_g(t,r)$ tends to a finite limit as $r\to\infty$. Hence $\psi_g(t,r)\in\Gamma(f)_r$.

We know, by [(3) (3, II)], that $\Gamma^*(f)=\sigma_1(f)$. Let U be a p-bd set in $\sigma_1(f)$; then, by [(5) (2, I) Corollary 1],

$$\int_0^\infty |h(r)|\,dr \leqslant K \tag{3.3}$$

for every $h(r)$ in U.

Then, by (3.2) and (3.3),

$$\left|\int_0^\infty \{\psi_g(t,r)-\psi_g(t',r)\}h(r)\,dr\right| \leqslant \epsilon\int_0^\infty |h(r)|\,dr \leqslant \epsilon K$$

for all t, $t'\geqslant T(\epsilon,g)$ and for every $h(r)$ in U.

Hence, by the definition of strong projective convergence [cf. (5)], $\psi_g(t,r)$ is $\mathbf{\Gamma(f)}_r$-cgt; and this is true for every $\beta(f)_x\alpha(f)$-(bd cgt) family $g(x,r)$. This establishes the theorem.

COROLLARY. *If $F(x,t)$ is $\sigma_\infty(f)_x$-cgt(s), then the F-transform of every $\sigma_1(f)_x$-(bd cgt) family $g(x,r)$ is $\mathbf{\Gamma(f)}_r$-cgt.*

(3, II). *If $F(x,t)$ is $\alpha(f)_x\beta(f)$-continuous(s), then the F-transform of every $\beta(f)_x\alpha(f)$-(bd cgt) family $g(x,r)$ is $\mathbf{\Gamma(f)}_r$-continuous.*

The proof is similar to that of (3, I).

(3, III). *If* (i) *$F(x,t)$ transforms $\alpha(f)$ into $\beta(f)$, and* (ii) *$\beta^*(f)\leqslant\sigma_1(f)$, then the F-transform of every $\alpha(f)_x$-cgt(s) family is $\beta(f)_t$-cgt.*

Let $g(x,r)$ be $\alpha(f)_x$-cgt(s), and let $\psi_g(t,r)$ be given by (3.1). By the hypothesis (i), $F(x,t)\in\alpha^*(f)_x$. Since $g(x,r)$ is $\alpha(f)_x$-cgt(s), to every $\epsilon>0$ there corresponds a positive number $R(\epsilon)$, independent of t, such that, by (3.2), for almost all $t\geqslant 0$,

$$|\psi_g(t,r)-\psi_g(t,r')| \leqslant \epsilon \tag{3.4}$$

for all r, $r'\geqslant R(\epsilon)$. By the hypothesis (i), $F(x,t)$ transforms $\alpha(f)$ into $\beta(f)$; therefore $\psi_g(t,r)\in\beta(f)_t$ for every $g(x,r)$ in $\alpha(f)_x$.

Let $h(t)$ be any function in $\beta^*(f)$; then, by (3.4) and hypothesis (ii),

$$\left|\int_0^\infty \{\psi_g(t,r)-\psi_g(t,r')\}h(t)\,dt\right| \leqslant \epsilon\int_0^\infty |h(t)|\,dt \leqslant \epsilon k(h)$$

for all r, $r'\geqslant R(\epsilon)$ for every $\epsilon>0$ and every $h(t)$ in $\beta^*(f)$.

Hence, by the definition of projective convergence [cf. (3)], $\psi_g(t,r)$

is $\beta(f)_t$-cgt, and this is true for every $\alpha(f)_x$-cgt(s) family $g(x,r)$. This proves the theorem.

It is observed that, if, in place of hypothesis (ii), we have

$$\beta^*(f) \geqslant \sigma_1(f),$$

then $\psi_g(t,r)$ is $\mathbf{\beta(f)_t\,\sigma_1(f)}$-cgt for every $\alpha(f)_x$-cgt(s) family $g(x,r)$.

(3, IV). *If* (i) $F(x,t)$ *transforms* $\alpha(f)$ *into* $\beta(f)$, *and* (ii) $\beta^*(f) \leqslant \sigma_1(f)$. *then the F-transform of every* $\alpha(f)_x$*-continuous(s) family is* $\beta(f)_t$*-continuous.*

The proof is similar to that of (3, III).

In this theorem also, if, in place of the hypothesis (ii), we postulate that $\beta^*(f) \geqslant \sigma_1(f)$, then the F-transform of every $\alpha(f)_x$-continuous(s) family $g(x,r)$ is $\mathbf{\beta(f)_t\,\sigma_1(f)}$-continuous.

(3, V). *If* (i) $\alpha^*(f) \geqslant \beta(f)$, (ii) $\alpha(f) \leqslant \sigma_1(f)$, *and* (iii) $F(x,t)$ *belongs to* $\alpha(f)_x$ *and is bounded in mean with exponent* 1, *then the F-transform of every parametric convergent family in* $\beta(f)$ *is parametric convergent.*

Let $g(x,r)$ belong to $\beta(f)_x$ and be parametric convergent (r-cgt), Since $g(x,r)$ is r-cgt, to every $\epsilon > 0$ there corresponds a positive number $R(\epsilon)$, independent of x, such that, for almost all $x \geqslant 0$,

$$|g(x,r)-g(x,r')| \leqslant \epsilon \tag{3.5}$$

for all r, $r' \geqslant R(\epsilon)$.

Since $F(x,t)$ is bounded in mean with exponent 1, we have, by (1.4 a),

$$\int_0^\infty |F(x,t)|\,dx \leqslant M \tag{3.6}$$

for all $t \geqslant 0$. Let $\psi_g(t,r)$ be given by (3.1); then, by (3.5) and (3.6), for all $t \geqslant 0$,

$$|\psi_g(t,r)-\psi_g(t,r')| = \left|\int_0^\infty F(x,t)\{g(x,r)-g(x,r')\}\,dx\right| \leqslant \epsilon M$$

for all r, $r' \geqslant R(\epsilon)$ for every $\epsilon > 0$. This proves the result.

(3, VI). *If* (i) *the kernel* $F(x,t)$ *belongs to* $\alpha(f)_x \leqslant \sigma_1(f)$ *and is bounded in mean with exponent* 1, (ii) $g(x,r)$ *belongs to* $\beta(f)_x \leqslant \alpha^*(f)$ *and is* r-cgt, (iii) $\psi_g(t,r)$ *is the F-transform of* $g(x,r)$ *and belongs to* $\gamma(f)_t$, *and* (iv) $\gamma^*(f) \leqslant \sigma_1(f)$, *then* $\psi_g(t,r)$ *belongs to* $\Gamma_0(f)_r$ *and is* $\gamma(f)_t$-cgt, *where* $\gamma(f)$ *is any function space.*

Since $F(x,t) \in \alpha(f)_x$, $g(x,r) \in \beta(f)_x$, and $\alpha^*(f) \geqslant \beta(f)$, $\psi_g(t,r)$, given by (3.1), is finite for all $t \geqslant 0$ and all $r \geqslant 0$.

By (ii), $g(x,r)$ is r-cgt; therefore given any $\epsilon > 0$, there exists a positive number $R(\epsilon)$, independent of x, such that (3.5) holds for all

$r, r' \geqslant R(\epsilon)$, and for almost all $x \geqslant 0$. By (i), $F(x,t)$ is bounded in mean with exponent 1, and therefore (3.6) holds for all $t \geqslant 0$.

Hence, by (3.5) and (3.6), we have, for all $t \geqslant 0$,

$$|\psi_g(t,r)-\psi_g(t,r')| = \left| \int_0^\infty F(x,t)\{g(x,r)-g(x,r')\}\,dx \right| \leqslant \epsilon M \qquad (3.7)$$

for all $r, r' \geqslant R(\epsilon)$. Therefore $\psi_g(t,r) \in \Gamma_0(f)_r$.

Let $h(t) \in \gamma^*(f)$; then, by (iii), (iv), and (3.7),

$$\left| \int_0^\infty \{\psi_g(t,r)-\psi_g(t,r')\}h(t)\,dt \right| \leqslant \epsilon M \int_0^\infty |h(t)|\,dt \leqslant \epsilon M'(h)$$

for all $r, r' \geqslant R(\epsilon)$ and for every $h(t)$ in $\gamma^*(f)$. Hence $\psi_g(t,r)$ is $\gamma(f)_t$-cgt.

We remark that, if in place of (iv) we postulate that $\gamma^*(f) \geqslant \sigma_1(f)$, then $\psi_g(t,r)$ is $\boldsymbol{\gamma(\mathfrak{f})_t\sigma_1(\mathfrak{f})}$-cgt.

'$f(x,t) \in \mathscr{C}_1(f)_t$' has the same meaning as '$f_t(x) \in \mathscr{C}_1(f)_t$', and for this we refer to (5).

(3, VII). *If the conditions* (i), (iii), *and* (iv) *of* (3, VI) *hold, but in place of* (ii) *we postulate that* $g(x,r) \in \beta(f)_x \leqslant \alpha^*(f)$ *and* $g(x,r) \in \mathscr{C}_1(f)_r$, *then* $\psi_g(t,r)$ *belongs to* $\mathscr{C}_1(f)_r$ *and is* $\gamma(f)_t$*-continuous.*

The proof is similar to that of (3, VI).

If, instead of (iv), we have $\gamma^*(f) \geqslant \sigma_1(f)$, then $\psi_g(t,r)$ is $\boldsymbol{\gamma(\mathfrak{f})_t\sigma_1(\mathfrak{f})}$-continuous.

(3, VIII). *If* (i) $F(x,t)$ *in* $\sigma_p(f)_x$ *is bounded in mean and continuous in mean with exponent* p, *and* (ii) $g(x,r)$ *in* $\sigma_q(f)_x$ *is bounded in mean and convergent in mean with exponent* q, *where* $1 \leqslant p \leqslant \infty$, *and*

$$1/p+1/q = 1,$$

then the F-transform of $g(x,r)$ *is*

(*a*) *r*-cgt, (*b*) $\boldsymbol{\mathscr{C}(\mathfrak{f})_t}$-cgt, *and* (*c*) $\boldsymbol{\Gamma(\mathfrak{f})_r}$*-continuous.*

Let us write $F_t(x) \equiv F(x,t)$, and $g_r(x) \equiv g(x,r)$. Let $\psi_g(t,r)$ be given by (3.1); then, by the conditions (i) and (ii) and the inequality (1.4), we have

$$|\psi_g(t,r)| = \left| \int_0^\infty F_t(x)g_r(x)\,dx \right| \leqslant \|F_t\|_p \cdot \|g_r\|_q \leqslant M \qquad (3.8)$$

for all $t \geqslant 0$ and all $r \geqslant 0$.

Again, by (i), (ii), (1.3), and (1.4),

$$\begin{aligned} |\psi_g(t,r)-\psi_g(t,r')| &= \left| \int_0^\infty F_t(x)\{g_r(x)-g_{r'}(x)\}\,dx \right| \\ &\leqslant \|F_t\|_p \cdot \|g_r-g_{r'}\|_q \leqslant k\epsilon \end{aligned} \qquad (3.9)$$

for all r, $r' \geqslant R(\epsilon)$ for every $\epsilon > 0$, and all $t \geqslant 0$. R is independent of t, so that $\psi_g(t,r)$ is r-cgt. This proves (a).

Also, by (i), (ii), (1.3), and (1.4), we have

$$|\psi_g(t,r)-\psi_g(t',r)| = \left| \int_0^\infty \{F_t(x)-F_{t'}(x)\}g_r(x)\,dx \right|$$

$$\leqslant \|F_t-F_{t'}\|_p \cdot \|g_r\|_q \leqslant \epsilon k' \qquad (3.10)$$

for all $r \geqslant 0$, and all non-negative t and t' such that $|t-t'| \leqslant \delta(\epsilon)$ for every $\epsilon > 0$. It follows from (3.8) and (3.9) that $\psi_g(t,r) \in \Gamma(f)_r$, and from (3.8) and (3.10) that $\psi_g(t,r) \in \mathscr{C}(f)_t$.

By [(**3**) (3, II)], $\mathscr{C}^*(f) = \Gamma^*(f) = \sigma_1(f)$.

Let U be a p-bd set in $\sigma_1(f)$; then, by [(**5**) (2, I) Corollary 1],

$$\int_0^\infty |h(t)|\,dt \leqslant L \qquad (3.11)$$

for every h in U. Then, by (3.9) and (3.11),

$$\left| \int_0^\infty \{\psi_g(t,r)-\psi_g(t,r')\}h(t)\,dt \right| \leqslant k\epsilon \int_0^\infty |h(t)|\,dt \leqslant \epsilon K$$

for all r, $r' \geqslant R(\epsilon)$ and for every h in U. Hence $\psi_g(t,r)$ is $\mathscr{C}(\mathbf{f})_t$-cgt. This proves (b).

Also, by (3.10) and (3.11),

$$\left| \int_0^\infty \{\psi_g(t,r)-\psi_g(t',r)\}h(r)\,dr \right| \leqslant \epsilon k' \int_0^\infty |h(r)|\,dr \leqslant \epsilon K'$$

for all non-negative t and t' such that $|t-t'| \leqslant \delta(\epsilon)$ and for every h in U. Hence $\psi_g(t,r)$ is $\mathbf{\Gamma}(\mathbf{f})_r$-continuous. This proves (c).

(3, IX). *If* (i) $F(x,t)$ *in* $\sigma_p(f)_x$ *is bounded in mean and convergent in mean with exponent* p, *and* (ii) $g(x,r)$ *in* $\sigma_q(f)_x$ *is bounded in mean and convergent in mean with exponent* q, *where* $1 \leqslant p \leqslant \infty$ *and* $1/p+1/q = 1$, *then the* F*-transform of* $g(x,r)$ *is* (a) r-cgt, (b) t-cgt, (c) $\mathbf{\Gamma}(\mathbf{f})_r$-cgt, *and* (d) $\mathbf{\Gamma}(\mathbf{f})_t$-cgt.

The proof is similar to that of (3, VIII).

(3, X). *If* (i) $F(x,t)$ *in* $\sigma_p(f)_x$ *is bounded in mean and continuous in mean with exponent* p, *and* (ii) $g(x,r)$ *in* $\sigma_q(f)_x$ *is bounded in mean and continuous in mean with exponent* q, *where* $1 \leqslant p \leqslant \infty$ *and* $1/p+1/q = 1$,

then the F-transform of $g(x,r)$ is (a) $\mathscr{C}(\mathfrak{f})_t$-continuous, and (b) $\mathscr{C}(\mathfrak{f})_r$-continuous.

The proof is similar to that of (3, VIII).

I wish to express my thanks to Dr. R. G. Cooke for useful suggestions.

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A GENERALIZATION OF THE HOPF CONSTRUCTION

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1. Introduction

I. M. JAMES has shown (**5**) that, with the exception of the toric constructions of H. Toda (**10**), the usual procedures for the construction of generators of homotopy groups of spheres give rise to no more elements than can be obtained by the Hopf construction together with the operation of composition. However, if $\alpha \in \pi_r(S^n)$ is obtainable by the Hopf construction, and if $w(\alpha)$ is the reduced product filtration of α [see (**8**)], then, by 13.10 of (**4**), $w(\alpha) \leqslant 2$. It is thus a problem to construct the generators whose filtrations exceed two and which are not known to be composition elements. Of these, many have been represented as toric constructions [see (**11**), (**12**)], but notably not, for example, the elements of order p in $\pi_{2p}(S^3)$ (p prime, $p > 3$). I shall prove in a subsequent paper that these latter are obtainable by the generalization of the Hopf construction which is presented here.

We denote the base-point of any space by $*$ and select the base-point of any topological product to be the point whose coordinates are the respective base-points. Let A be a *special* complex: that is, a countable CW-complex with precisely one 0-cell, namely $*$. Then A_∞, the reduced product complex of A [see (**3**)], is a countable CW-complex whose points are equivalence classes of finite sequences of points of A. Let A_m ($m \geqslant 0$) be the subcomplex determined by the sequences with m terms or fewer, A_0 being identified with $*$ and A_1 with A. Let M be the topological product $S^{r_1} \times S^{r_2} \times \ldots \times S^{r_j}$. Given a map

$$F\colon M \to A_\infty \tag{1.1}$$

which respects base-points, by the *i-th section* of F we mean the map $f_i\colon S^{r_i} \to A_\infty$ such that

$$f_i(y_i) = F(y_1, y_2, \ldots, y_j),$$

where $y_k \in S^{r_k}$ and $y_k = *$ if $k \neq i$. If $f_i(S^{r_i}) \subseteq A_{m_i}$, let $\alpha_i \in \pi_{r_i} A_{m_i}$ be the element defined by f_i. Then, if $F(M) \subseteq A_k$, we describe F as a map of *type*

$$(\alpha_1, \alpha_2, \ldots, \alpha_j)^k. \tag{1.2}$$

Quart. J. Math. Oxford (2), 12 (1961), 196-204.

Let $F'\colon M \to A_\infty$ be the map such that

$$F'(y_1, y_2, \ldots, y_j) = f_1(y_1) \cdot f_2(y_2) \ldots f_j(y_j), \tag{1.3}$$

where the fullpoint denotes the multiplication in A_∞. Then, if $m = \sum m_i$, F' is a map of type $(\alpha_1, \alpha_2, \ldots, \alpha_j)^m$. If $F(M) \subseteq A_k$ and in addition F agrees with F' except on the $(\sum r_i)$-cell of M, we describe F as a map *strongly of type*

$$(\alpha_1, \alpha_2, \ldots, \alpha_j)^k. \tag{1.4}$$

If X and Y are spaces, we denote the set of homotopy classes of maps $X \to Y$ by $[X, Y]$, the collapsed product of X and Y by $X \wedge Y$, the (reduced) suspension of X by sX and, if $f\colon X \to Y$ is a map, we denote the suspension of f by $s(f)$. Let ${}_1P$, ${}_2P, \ldots,$ ${}_jP$ be special complexes. D. Puppe (9) has proved that $s({}_1P \times {}_2P \times \ldots \times {}_jP)$ has the homotopy type of a certain wedge W (say) of suspensions of collapsed sub-products of ${}_1P \times {}_2P \times \ldots \times {}_jP$. We shall construct a particular homotopy equivalence

$$\theta\colon s({}_1P \times {}_2P \times \ldots \times {}_jP) \to W \tag{1.5}$$

[by the method due to James in the proof of 4.1 of (6)], which is the track sum in a certain order of the suspensions of the projections on to the collapsed sub-products. Then W contains precisely one copy of $Z = s({}_1P \wedge {}_2P \wedge \ldots \wedge {}_jP)$. Let ι be the homotopy class of the injection map $Z \to W$. Let Ω be the space of loops on sA and let $\phi_1\colon A_\infty \to \Omega$ be a canonical map [see (3)]. Let $\psi\colon sA_\infty \to sA$ be the map such that $\psi\nu(x, t) = \phi_1 x(t)$, where $x \in A_\infty$, $t \in I$, and $\nu\colon A_\infty \times I \to sA_\infty$ is the suspension identification. Then, if $F\colon {}_1P \times {}_2P \times \ldots \times {}_jP \to A_\infty$ is a based map, we define

$$c(F) = \psi_*\{s(F)\}_*(\theta_*)^{-1}\iota \in [Z, sA] \tag{1.6}$$

to be the element obtained from F by the *generalized Hopf construction*. It should be emphasized that the homotopy class of θ, and hence $c(F)$, depends on the chosen order of track addition. While we shall not study the effect of varying the order, we defend our choice to some extent by proving that in the case ${}_kP = A_{m_k}$ $(k = 1, 2, \ldots, j)$ the diagram

$$\begin{array}{ccc} s(A_{m_1} \times A_{m_2} \times \ldots \times A_{m_j}) & \xrightarrow{s(p)} & s(A_\infty) \\ \downarrow{\scriptstyle \theta} & & \downarrow{\scriptstyle \psi} \\ W & \xrightarrow{\quad\sigma\quad} & sA \end{array} \tag{1.7}$$

is homotopy-commutative, where p is the natural map [see (3) 176] and σ is the map which agrees with ψ on $s(A_{m_k})$ $(k = 1, 2, \ldots, j)$ and sends the rest of W to $* \in sA$.

Let $\phi\colon \pi_r(A_\infty) \to \pi_{r+1}(sA)$ be the canonical isomorphism of (3) and let

$F\colon M \to A_\infty$ be a map strongly of some type. Then the separation element $d(F', F) \in \pi_r(A_\infty)$ $(r = \sum r_i)$ is defined [see (10) of (7)]. The main result (orientations being chosen in a standard manner) is the theorem:

THEOREM 1.8. $\quad c(F) = (-1)^{r+1} d(F', F).$

It can be verified by comparing† Theorem 1.8 with 5.7 of (7) that we recover the classical Hopf construction if $j = m = 2$. The following, which is easily deduced from Theorem 1.8 and the properties of separation elements, reduces to a result by James if $j = m = 2$:

THEOREM 1.9. *If F is a map strongly of type $(\alpha_1, \alpha_2, \ldots, \alpha_j)^{m-1}$ such that $c(F) = \alpha$, then there exists a map F'' strongly of the same type such that $c(F'') = \alpha''$ if and only if $w(\alpha - \alpha'') \leqslant m-1$.*

Let $A_{(m)}$ be the m-fold collapsed product of A with itself. Then the map $h_m\colon A_m \to A_{(m)}$ which shrinks A_{m-1} to $*$ is an identification. We shall also denote by h_m the induced homomorphism $\pi_r(A_m) \to \pi_r(A_{(m)})$. Let $A(m) = sA_{(m)}$ and let

$$H_m\colon \pi_r(sA) \to \pi_r(A(m)) \quad (m = 1, 2, \ldots)$$

be the Hopf–James invariants [see (2) for the case $A = S^n$]. Let $\alpha \wedge \beta$ denote the collapsed product of the homotopy classes α, β and let E denote the suspension homomorphism. Then we shall prove the following generalization of the classical result concerning the Hopf invariant of a Hopf construction:

THEOREM 1.10. *If $\alpha_i \in \pi_{r_i}(A_{m_i})$ $(i = 1, 2, \ldots, j)$, $m = \sum m_i$, and F is a map strongly of type $(\alpha_1, \alpha_2, \ldots, \alpha_j)^{m-1}$, then*

$$H_m c(F) = -E(h_{m_1}\alpha_1 \wedge h_{m_2}\alpha_2 \wedge \ldots \wedge h_{m_j}\alpha_j).$$

Let α, ι_2 be generators of $\pi_5(S^2)$, $\pi_2(S^2)$ respectively. I shall prove in the paper to follow that there exists a map F strongly of type $(\alpha, \iota_2, \iota_2, \iota_2)^3$. Then Theorem 1.10 implies that $c(F)$ is a non-zero element of $\pi_{12}(S^3)$ and $w(c(F)) = 4$.

2. The homotopy equivalence

Let I denote the closed unit interval, S^1 the space obtained from I by shrinking $(0) \cup (1)$ to $*$ and let $f \wedge g\colon X \wedge Y \to Z \wedge W$ denote the collapsed product of the based maps $f\colon X \to Z$, $g\colon Y \to W$. We define sX, the (reduced) *suspension* of X, to be $X \wedge S^1$ and $s(f)\colon sX \to sY$, the *suspension* of $f\colon X \to Y$, to be the map $f \wedge 1\colon X \wedge S^1 \to Y \wedge S^1$. If

† See also *J. London Math. Soc.* **35** (1960) 480, footnote.

$x \in X$, $y \in Y$, we denote the images of $(x, y) \in X \times Y$, $t \in I$, $(x, t) \in X \times I$ under the corresponding identifications by $x \wedge y \in X \wedge Y$, $t \in S^1$, $(x, t) = x \wedge t \in sX$. Let ${}_if\colon sX \to Y$ $(i = 1, 2, \ldots, n)$ be based maps. Then we denote by

$$\sum_{i=1}^{n} {}_if\colon sX \to Y \tag{2.1}$$

the map such that

$$\sum_{i=1}^{n} {}_if(x, t) = {}_if(x, nt - i + 1) \quad \text{if} \quad i - 1 \leqslant nt \leqslant i \quad (i = 1, 2, \ldots, n).$$

If $+$ denotes the track addition [see (1)] in $[sX, Y]$, we observe that

$$\left\{\sum_{i=1}^{n} {}_if\right\} = ((\ldots(\{{}_1f\} + \{{}_2f\}) + \{{}_3f\}) + \ldots) + \{{}_nf\}) = \sum_{i=1}^{n} \{{}_if\} \quad \text{(say)}.$$

Let $\mathscr{P}$ denote the ordered set $\{{}_rP\colon r = 1, 2, \ldots, j\}$ of disjoint special complexes, $\Pi\mathscr{P}$ the topological product ${}_1P \times {}_2P \times \ldots \times {}_jP$, and $\Delta\mathscr{P}$ the collapsed product ${}_1P \wedge {}_2P \wedge \ldots \wedge {}_jP$. Let i be an integer, let c denote the binomial coefficient $\binom{j}{i}$, and let $\{\epsilon(k)\colon k = 1, 2, \ldots, c\}$ be the set of increasing functions from $(1, 2, \ldots, i)$ into $(1, 2, \ldots, j)$ ordered *lexicographically from the right*. This means that, if $p < q \leqslant c$, then there exists r such that $\epsilon(p)(t) = \epsilon(q)(t)$ for $r < t \leqslant i$ and $\epsilon(p)(r) < \epsilon(q)(r)$. We denote the ordered set

$$\{{}_rP\colon r = \epsilon(k)(1), \epsilon(k)(2), \ldots, \epsilon(k)(i)\}$$

by $\mathscr{P}_{i,k}$. Let $\pi_{i,k}\colon \Pi\mathscr{P} \to \Pi\mathscr{P}_{i,k}$ be the projection which suppresses the coordinates corresponding to spaces which are not members of $\mathscr{P}_{i,k}$ and let $\mu\colon \Pi\mathscr{P}_{i,k} \to \Delta\mathscr{P}_{i,k}$ be the collapsing map. Let $W = W(\mathscr{P})$ be the wedge with common point $*$ of the complexes $s\Delta\mathscr{P}_{i,k}$, k running from 1 to c, and i from 1 to j. Let $\lambda_{i,k}\colon s\Delta\mathscr{P}_{i,k} \to W$ be the injection and let $\theta\colon s\Pi\mathscr{P} \to W$ be the map

$$\theta = \sum_{i=1}^{j} \sum_{k=1}^{c} \lambda_{i,k}\, s(\mu\pi_{i,k}). \tag{2.2}$$

Then we have the theorem:

THEOREM 2.3. *$\theta\colon s\Pi\mathscr{P} \to W(\mathscr{P})$ is a homotopy equivalence.*

Proof. Since $s\Pi\mathscr{P}$ and W are simply-connected CW-complexes, it will be sufficient by Theorem 3 of (13) to prove that the induced homomorphism $\theta_*\colon H.(s\Pi\mathscr{P}) \to H.(W)$ of the singular integral homology groups is an isomorphism. The case $j = 2$ follows by an argument using the homology sequence of the pair $({}_1P \times {}_2P, {}_1P \vee {}_2P)$. While $[s\Pi\mathscr{P}, W]$ is not necessarily abelian, it is important to observe that θ_* does not depend on the order of the addition in (2.2). Let $\mathscr{P}'$ be the

ordered set $\{{}_rP, r = 2,3,...,j\}$ and let W' and θ': $s\mathscr{P}' \to W' = W(\mathscr{P}')$ be defined correspondingly. Then we can argue inductively for the composition of the homotopy equivalence

$$s\Pi\mathscr{P} = \Pi\mathscr{P} \wedge S^1 \to ({}_1P \wedge S^1) \vee (\Pi\mathscr{P}' \wedge S^1) \vee ({}_1P \wedge \Pi\mathscr{P}' \wedge S^1)$$

with the wedge of the equivalences $1: {}_1P \wedge S^1 \to {}_1P \wedge S^1$, θ'; and $1 \wedge \theta'$: ${}_1P \wedge \Pi\mathscr{P}' \wedge S^1 \to {}_1P \wedge W'$ turns out to be the required track sum (but in a different order).

Let $\mathscr{M} = \{{}_rM, r = 1,2,...,j\}$ be an ordered set of special complexes and let ${}_rg$: ${}_rM \to {}_rP$ $(r = 1,2,...,j)$ be based maps. Then, if g is the ordered set $\{{}_rg, r = 1,2,...,j\}$, we denote by Πg: $\Pi\mathscr{M} \to \Pi\mathscr{P}$ the product map ${}_1g \times {}_2g \times \ldots \times {}_jg$ and by Δg: $\Delta\mathscr{M} \to \Delta\mathscr{P}$ the collapsed product map ${}_1g \wedge {}_2g \wedge \ldots \wedge {}_jg$. Let

$$w(g)\colon W(\mathscr{M}) \to W(\mathscr{P}) \tag{2.4}$$

be the map such that $w(g) \mid s\Delta\mathscr{M}_{i,k} = s(\Delta g_{i,k})$. Then the following diagram is commutative.

$$\begin{array}{ccc} s\Pi\mathscr{M} & \xrightarrow{s(\Pi g)} & s\Pi\mathscr{P} \\ \downarrow{\scriptstyle\theta} & & \downarrow{\scriptstyle\theta} \\ W(\mathscr{M}) & \xrightarrow{w(g)} & W(\mathscr{P}) \end{array} \tag{2.5}$$

We can deduce from (1.6) and (2.5) the theorem:

THEOREM 2.6. $c(F\Pi g) = c(F) \circ \{\Delta g\} \in [s\Delta\mathscr{M}, sA]$.

3. A commutative diagram

Let $\mathscr{A}$ be the ordered set $\{A_m, i = 1,2,...,j\}$, let q_i: $A_{m_i} \to A_\infty$ be the injection map and let p: $\Pi\mathscr{A} \to A_\infty$ be the natural map. Then we have the theorem:

THEOREM 3.1. $\{\psi s(p)\} = \sum_{i=1}^{j} \{\psi s(q_i \pi_{1,i})\} \in [s\Pi\mathscr{A}, sA]$.

Proof. Let $m' = m_{j-1} + m_j$. Then, since p can be factored into maps $\Pi\mathscr{A} \to A_{m_1} \times \ldots \times A_{m_{j-1}} \times A_{m'} \to A_\infty$, an inductive argument using the left distributivity of the composition operation for suspension classes reduces the problem to the case $j = 2$. Let $m_1 = m$, $m_2 = n-m$, $t \in I$, and let

$$x = (x_1 x_2 \ldots x_m, x_{m+1} x_{m+2} \ldots x_n), \qquad x \in A_m \times A_{n-m}.$$

Let ϕ_1: $A_\infty \to \Omega$ be the canonical map associated with a distance η_1 relative to $*$ in A. Let

$$\alpha_k = \eta_1 x_k, \qquad \alpha(k) = \alpha_1 + \alpha_2 + \ldots + \alpha_k \quad (k = 1,2,...,n),$$
$$\beta = \alpha(n) - \alpha(m).$$

Then $$\psi s(p)(x,t) = \phi_1(x_1 x_2 \dots x_n)(t),$$

and referring to the definition of the canonical map [(3) 190] we can verify that

$$\psi s(p)(x,t)$$

$$= \begin{cases} \psi(q_1\pi_{1,1}x,\ t\alpha(n)/\alpha(m)) & \text{if } \alpha(m) \neq 0 \\ * & \text{if } \alpha(m) = 0 \end{cases} \Big\} \text{if } 0 \leqslant t\alpha(n) \leqslant \alpha(m),$$

$$\begin{cases} \psi(q_2\pi_{1,2}x,\ (t\alpha(n)-\alpha(m))/\beta) & \text{if } \beta \neq 0 \\ * & \text{if } \beta = 0 \end{cases} \Big\} \text{if } \alpha(m) \leqslant t\alpha(n) \leqslant \alpha(n),$$

while

$$(\psi s(q_1\pi_{1,1}) + \psi s(q_2\pi_{1,2}))(x,t) = \begin{cases} \psi(q_1\pi_{1,1}x,\ 2t) & \text{if } 0 \leqslant t \leqslant \tfrac{1}{2}, \\ \psi(q_2\pi_{1,2}x,\ 2t-1) & \text{if } \tfrac{1}{2} \leqslant t \leqslant 1. \end{cases}$$

Since $x \to \alpha(n)$, $x \to \alpha(m)$ are continuous functions, we can construct the required homotopy by a standard technique. The only difficulty which arises is the question of its continuity at points (x,t) for which $\alpha(m) = 0$ or $\beta = 0$. At such points we have $\pi_{1,1}x = *$ or $\pi_{1,2}x = *$ (respectively) and the continuity there is a consequence of the Lemma 3.2 following.

If X is any space, let $\nu\colon X \times I \to sX$ be the suspension identification. Then ν shrinks $(*) \times I \cup X \times (0) \cup X \times (1)$ to $* \in sX$.

LEMMA 3.2. *If U is any open set of sX containing $*$, then there exists an open set $U' \subseteq X$ such that $* \in U'$ and $\nu(U' \times I) \subseteq U$.*

Since Lemma 3.2 follows easily from the compactness of I, the proof of Theorem 3.1 is complete.

From (2.2) and Theorem 3.1 we deduce the corollary:

COROLLARY 3.3. *Diagram* 1.7 *is homotopy-commutative.*

Let ${}_if\colon {}_iP \to A_\infty$ be based maps ($i = 1, 2, \dots, j$) and let $F'\colon \Pi\mathscr{P} \to A_\infty$ be the map such that

$$F'(y_1, y_2, \dots, y_j) = {}_1f(y_1) \cdot {}_2f(y_2) \cdot \dots \cdot {}_jf(y_j).$$

We have the further corollary:

COROLLARY 3.4. *$c(F')$ is the class of the constant map $s\Delta\mathscr{P} \to sA$.*

We first observe that the method of constructing the homotopy is such that Corollary 3.3 remains valid if each A_m in $\mathscr{A}$ is replaced by A_∞. Then Theorem 2.6 implies that $p\colon \Pi\mathscr{A} \to A_\infty$ is a 'universal example'. Applying Corollary 3.3 we have $c(p) = \sigma_* \iota$, which is the class of the constant map.

4. The separation element

In this section we prove Theorems 1.8 and 1.10. Let V^n and S^n denote the n-element and n-sphere as represented in (4). We shall

assume that orientations for V^n, S^n, products and collapsed products of spheres have been chosen as in (4) and we observe that these are consistent with our earlier representation of sS^n as $S^n \wedge S^1$ provided that the map $\phi_1 : I \to S^1$ [(4) 201] is chosen as the identification.

Let K be a CW-complex whose characteristic maps respect base-points and let L be a subcomplex such that $K - L = \mathscr{E}^r$, an open r-cell ($r \geqslant 1$). We may represent sK as the CW-complex whose characteristic maps are the suspensions of the characteristic maps of K. Let $u, v : K \to X$ be maps which agree on L. Then the separation element $d(s(u), s(v)) \in \pi_{r+1}(sX)$ is defined. We need the following lemma:

LEMMA 4.1. $\qquad d(s(u), s(v)) = -Ed(u, v).$

Proof. Let $f : V^r,\ S^{r-1} \to K,\ L$ be a characteristic map for $\mathscr{E}^r$. Let $p_r, q_r : V^r \to S^r$ denote the projections orthogonal to the hyper-plane of V^r on to the 'northern' and 'southern' hemispheres of S^r respectively. Then, by 10.1 of (7), $d(u, v)$ and $d(s(u), s(v))$ are the homotopy classes of maps

$$g : S^r \to X, \qquad g' : S^{r+1} \to sX$$

such that

$$gp_r = uf, \qquad gq_r = vf, \qquad g'p_{r+1} = s(uf), \qquad g'q_{r+1} = s(vf).$$

Then applying 1.1 and 1.2 of (4) we can verify that

$$g'((x, s), t) = s(g)((x, t), s),$$

where $x \in S^{r-1}$, $s \in I$, $t \in I$. This implies Lemma 4.1.

We can represent S^r, $r \geqslant 1$, as a CW-complex by choosing, as characteristic map for the r-cell, the map $\psi_r : V^r,\ S^{r-1} \to S^r,\ *$ of (4). Let $u, v : S^r, * \to X, *$ be two maps. We have the lemma:

LEMMA 4.2. $\qquad (-1)^r d(u, v) = \{u\} - \{v\} \in \pi_r(X, *).$

Proof. Now $d(u, v)$ is the homotopy class of a map $g : S^r \to X$ such that $gp_r = u\psi_r$, $gq_r = v\psi_r$. Applying 1.1 and 1.3 of (4) we find that

$$g(x, t) = \begin{cases} v\lambda(x, 2t) & \text{if } 0 \leqslant t \leqslant \frac{1}{2}, \\ u\lambda(x, 2-2t) & \text{if } \frac{1}{2} \leqslant t \leqslant 1, \end{cases}$$

where $x \in S^{r-1}$ and $\lambda : S^r \to S^r$ is a map of degree $(-1)^{r+1}$. Lemma 4.2 follows by the definition of addition.

Proof of Theorem 1.8. Let ${}_iP = S^{r_i}$ $(i = 1, 2, \ldots, j)$ and let

$$F : M = \Pi\mathscr{P} \to A_\infty$$

be a map strongly of some type. Let $\tau' : W = W(\mathscr{P}) \to sM$ be a homo-

topy inverse of θ which respects base-points and let τ be the restriction of τ' to $S^{r+1} = s\Delta\mathscr{P} \subseteq W$. Let $\mathscr{E}$ be the $(r+1)$-cell of sM. Then, since $\theta\colon sM \to W$ preserves orientations, the map $\tau\colon S^{r+1}, * \to sM, sM - \mathscr{E}$ is of degree 1. It is easily verified that $\phi = \psi_* E\colon \pi_r(A_\infty) \to \pi_{r+1}(sA)$. Thus we have

$$\begin{aligned}
\phi d(F', F) &= \psi_* E d(F', F) \\
&= -\psi_* d(s(F'), s(F)), \qquad \text{by Lemma 4.1,} \\
&= -\psi_* d(s(F')\tau, s(F)\tau), \quad \text{by 10.9 of (7),} \\
&= (-1)^r(\{\psi s(F')\tau\} - \{\psi s(F)\tau\}), \quad \text{by Lemma 4.2.}
\end{aligned}$$

Since however $\{\psi s(F')\tau\} = c(F')$ and $\{\psi s(F)\tau\} = c(F)$, Theorem 1.8 follows from Corollary 3.4.

Proof of Theorem 1.10. Let $h\colon A_\infty \to (A_{(m)})_\infty$ be the combinatorial extension [see (3)] of h_m. By (1.6) and the definition of the Hopf–James invariants,

$$H_m c(F) = (-1)^{r+1}\phi h_* d(F', F) = (-1)^{r+1}\phi d(hF', hF).$$

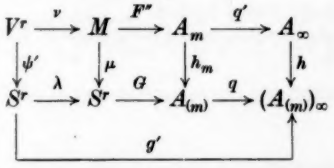
Let $\nu\colon V^r, S^{r-1} \to M, M - \mathscr{E}^r$ be an orientation-preserving characteristic map for the r-cell $\mathscr{E}^r$ of M. Then $-d(hF', hF)$ has a representative map $g\colon S^r \to (A_{(m)})_\infty$ such that $gp_r = hF\nu$ and $gq_r = hF'\nu$. Since F is a map of type $(\alpha_1, \alpha_2, \ldots, \alpha_j)^{m-1}$, then by 1.1 of (4) and in the notation of (4),

$$g(x,t) = \begin{cases} * & \text{if } \frac{1}{2} \leqslant t \leqslant 1, \\ hF'\nu(2t, x) & \text{if } 0 \leqslant t \leqslant \frac{1}{2}. \end{cases}$$

Thus g is homotopic to a map g' such that

$$g'(x,t) = hF'\nu(t,x). \tag{4.3}$$

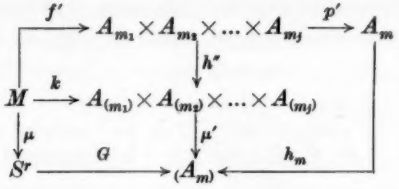
We have the commutative diagram



where F'' agrees with F', q and q' are injections, μ is the collapsed product identification, ψ' is the map on [(4) 194], λ is a map of degree $(-1)^{r+1}$, and $G = h_m F''\mu^{-1}$. Thus

$$H_m c(F) = (-1)^r\phi\{g'\} = -\phi q_*\{G\} = -E\{G\},$$

by 10.2 of (4). Let $f_i\colon S^{r_i} \to A_{m_i}$ $(i = 1, 2, \ldots, j)$ agree with the sections of F. Then Theorem 1.10 follows by the commutativity of the diagram



where

$$f' = f_1 \times f_2 \times \ldots \times f_j, \qquad h'' = h_{m_1} \times h_{m_2} \times \ldots \times h_{m_j}, \qquad k = h''f',$$

μ and μ' are collapsed product identifications and where p' agrees with the natural map, so that $F' = p'f'$.

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FINITE PRESENTATIONS OF GROUPS AND 3-MANIFOLDS

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THE deficiency of a finite presentation of a group is the number of generators minus the number of relations in the presentation. It has been known for some time that there is a connexion between the deficiency and the second homology group of G (**6**, **4**). In this paper an inequality is proved which includes much of the known information in this direction. This inequality is essentially due to Professor P. Hall, whose help the author is glad to acknowledge.

§ 1 consists of group theory only. In § 2 and § 3, the results of § 1 are applied to the fundamental groups of 3-manifolds.

The author would like to thank Dr. E. C. Zeeman for his constant encouragement, and for simplifying one of the proofs.

1. An upper bound for the deficiency

Let $P = \{x_1,..., x_n/r_1,..., r_m\}$ be the presentation of a group G. Then

$$\text{deficiency of } P = \operatorname{def} P = n-m.$$

We define deficiency of $G = \operatorname{def} G =$ maximum of $\operatorname{def} P$ over all possible presentations P of G.

We shall see from Lemma 1.2 that $\operatorname{def} G$ always exists if G is finitely presentable.

Let $s(G)$ be the minimum number of generators of G. Let F be the free group on the n generators $x_1,..., x_n$. Let R be the normal subgroup of F, generated by $r_1,..., r_m$. Then $G \cong F/R$. We have the lemma:

LEMMA 1.1. *$R/[F, R]$ is an abelian group generated by the m generators $r_1[F, R],..., r_m[F, R]$.*

$R/[F, R]$ is abelian since $[R, R] \subset [F, R]$. Any element of R is of the form

$$f_1^{-1} r_{i_1}^{\epsilon_1} f_1 \cdots f_s^{-1} r_{i_s}^{\epsilon_s} f_s,$$

where $f_j \in F$ and $\epsilon_j = \pm 1$ $(1 \leqslant j \leqslant s)$. So Lemma 1.1 follows. We have a second lemma:

LEMMA 1.2. $\operatorname{def} P \leqslant \operatorname{rank} H_1(G; Z) - s(H_2(G; Z))$.

We have the following diagram of abelian groups and homomorphisms, in which the rows are exact

$$0 \longrightarrow [F, F] \cap R/[F, R] \longrightarrow R/[F, R] \longrightarrow R/[F, F] \cap R \longrightarrow 0$$

$$0 \longleftarrow F/[F, F]R \longleftarrow F/[F, F] \longleftarrow [F, F]R/[F, F] \longleftarrow 0.$$

By a well-known result due to Hopf (**2**),

$$H_2(G; Z) \cong [F, F] \cap R/[F, R].$$

Also $$H_1(G; Z) \cong G/[G, G] \cong F/[F, F]R.$$

So the diagram becomes

$$0 \longrightarrow H_2(G; Z) \longrightarrow R/[F, R] \longrightarrow A \longrightarrow 0$$

$$0 \longleftarrow H_1(G; Z) \longleftarrow F/[F, F] \longleftarrow A \longleftarrow 0.$$

Since $F/[F, F]$ is free abelian, so is A. Therefore the first exact sequence splits and

$$R/[F, R] \cong H_2(G; Z) + A.$$

Since A is free abelian,

$$s(R/[F, R]) = s(H_2(G; Z)) + s(A).$$

But $s(R/[F, R]) \leqslant m$ by (1.1) and

$$s(A) = n - \operatorname{rank} H_1(G; Z)$$

by the second exact sequence. Therefore

$$s(H_2(G; Z)) + n - \operatorname{rank} H_1(G; Z) \leqslant m.$$

Then Lemma 1.2 follows.

If G is a group such that equality is attained in Lemma 1.2, we say that G is *efficient*. The author suspects, but cannot prove, that $(Z \times Z_2) * (Z \times Z_3)$ is not efficient. We have the lemma:

LEMMA 1.3. *If G is a finitely generated abelian group, it is efficient.*

We simply use the canonical presentation of G.

Let $\operatorname{rank}_p V$ denote the dimension of the vector space V over the field Z_p, where p is a prime. Then we have the lemma:

LEMMA 1.4. *If X is a free algebraic complex, finitely generated in dimensions* 1 *and* 2, *then*

$$\operatorname{rank}_p H_1(X; Z_p) - \operatorname{rank}_p H_2(X; Z_p) \geqslant \operatorname{rank} H_1(X; Z) - s(H_2(X; Z)).$$

We have inequality if and only if the canonical form

$$H_2(X; Z) \cong Z + \ldots + Z + Z_{i_1} + \ldots + Z_{i_k} \quad (i_r \mid i_{r+1},\ 1 \leqslant r < k)$$

is such that $k > 0$ and $p \mid i_1$.

Let $H = \mathrm{Tor}(H_1(X;Z), Z_p)$. Then, by the universal coefficient theorem,

$$H_1(X;Z_p) \cong H_1(X;Z) \otimes Z_p \cong (\mathrm{rank}\, H_1(X;Z))Z_p + H,$$

$$H_2(X;Z_p) \cong H_2(X;Z) \otimes Z_p + H \cong rZ_p + H,$$

where

$$r < s(H_2(X;Z)) \quad (p \nmid i_1),$$

$$r = s(H_2(X;Z)) \quad (p \mid i_1).$$

Therefore

$$\mathrm{rank}_p H_1(X;Z_p) = \mathrm{rank}\, H_1(X;Z) + \mathrm{rank}_p H,$$

$$\mathrm{rank}_p H_2(X;Z_p) = r + \mathrm{rank}_p H.$$

Therefore

$$\mathrm{rank}_p H_1(X;Z_p) - \mathrm{rank}_p H_2(X;Z_p) = \mathrm{rank}\, H_1(X;Z) - r$$
$$\geqslant \mathrm{rank}\, H_1(X;Z) - s(H_2(X;Z)).$$

Thus Lemma 1.4 follows. We have further lemmas:

LEMMA 1.5. $\mathrm{rank}_p H_1(G;Z_p) - \mathrm{rank}_p H_2(G;Z_p) \geqslant \mathrm{def}\, G$.

This follows from Lemmas 1.4 and 1.2.

We say that G is *p-efficient* if there is equality in Lemma 1.5. If G is p-efficient, it is efficient. If G is efficient, then, by Lemma 1.4, it is p-efficient for some p.

LEMMA 1.6. *If A and B are p-efficient groups, then $A * B$ is p-efficient and* $\mathrm{def}\, A * B = \mathrm{def}\, A + \mathrm{def}\, B$.

For

$$\mathrm{rank}_p H_1(A;Z_p) + \mathrm{rank}_p H_1(B;Z_p) - \mathrm{rank}_p H_2(A;Z_p) - \mathrm{rank}_p H_2(B;Z_p)$$
$$= \mathrm{rank}_p H_1(A*B;Z_p) - \mathrm{rank}_p H_2(A*B;Z_p)$$
$$\geqslant \mathrm{def}\, A*B \geqslant \mathrm{def}\, A + \mathrm{def}\, B \text{ (this follows by adjoining a presentation of } A \text{ to a presentation of } B)$$
$$= \mathrm{rank}_p H_1(A;Z_p) + \mathrm{rank}_p H_1(B;Z_p) - \mathrm{rank}_p H_2(A;Z_p) - \mathrm{rank}_p H_2(B;Z_p).$$

So Lemma 1.6 follows.

LEMMA 1.7. *If G has a presentation with only one non-trivial relation $\{x_1,\ldots,x_n/r\}$, then G is efficient and* $\mathrm{def}\, G = n-1$.†

In the proof of Lemma 1.2, the only place where equality between $n-1$ and $\mathrm{rank}\, H_1(G;Z) - s(H_2(G;Z))$ could break down is in Lemma

† The author wishes to thank C. T. C. Wall for his assistance in proving this lemma.

1.1: that is, we have equality in 1.2 unless $R/[R, F]$ has 0 generators. If $R/[R, F] = 0$, then $R = [R, F]$. By induction,

$$R = [[...[[R, F], F],..., F], F] \leqslant [[... [[F, F], F],..., F], F].$$

So
$$R \leqslant \bigcap [[... [[F, F], F],..., F], F],$$

where we take the intersection over all possible subgroups of the form given. By [(3) 38], $R = 0$. Therefore $r = 1$. But we have supposed r non-trivial. Therefore $s(R/[F, R]) = 1$, and Lemma 1.7 follows.

2. The fundamental group of a 3-manifold

All manifolds appearing in this paper will be connected, compact and triangulated. They may have boundaries. We have the lemmas:

LEMMA 2.1. *If M is an n-manifold, there is a cell-decomposition of M, with only one n-cell, and with a simplicial $(n-1)$-skeleton.*

We simply abolish, one by one, interiors of $(n-1)$-simplexes in the triangulation of M.

LEMMA 2.2. *If M is a 3-manifold with boundary, then*

$$\operatorname{def} \pi_1(M) \geqslant 1-\chi(M).$$

We deformation-retract M by pushing in the interior of one 2-simplex of $\operatorname{Bd} M$ until the 3-cell of Lemma 2.1 is entirely retracted into the 2-skeleton of M. Let there now be p_n n-simplexes. A maximal tree contains (p_0-1) 1-simplexes. There is therefore a presentation of $\pi_1(M)$ with p_1-p_0+1 generators and p_2 relations. Therefore

$$\operatorname{def} \pi_1(M) \geqslant p_1-p_0+1-p_2 = 1-\chi(M).$$

LEMMA 2.3. *If M is a closed 3-manifold, then* $\operatorname{def} \pi_1(M) \geqslant 0$.

We remove a small open 3-ball from M, obtaining a 3-manifold M' with boundary. Now $\chi(M) = 0$ by Poincaré duality. Therefore $\chi(M') = 1$. Also $\pi_1(M') \cong \pi_1(M)$. Therefore

$$\operatorname{def} \pi_1(M) = \operatorname{def} \pi_1(M') \geqslant 1-\chi(M') = 0.$$

LEMMA 2.4. *If M is an n-manifold, there is at most one torsion element in $H_{n-1}(M; Z)$, and this element (if it exists) is of order two.*

We use the cell decomposition of Lemma 2.1. By its construction, the boundary of the n-cell is either zero, or twice the union of certain $n-1$ simplexes (with appropriate orientations). Lemma 2.4 follows.

We now use the result [(2.1) of (**1**)]:

PROJECTIVE PLANE THEOREM. *If M is a (compact) 3-manifold, there exists a finite set γ of maps g_α: $S^2 \longrightarrow M-\operatorname{Bd} M$, which are a set of $\pi_1(M)$-generators of $\pi_2(M)$. Each map g_α is semilinear. The images of*

g_α and g_β do not intersect if $\alpha \neq \beta$. Each map g_α is either non-singular, or identifies antipodal points. The image $g_\alpha(S^2)$ is a 2-sphere or a two-sided projective plane in M.

Let $n(M, \gamma)$ be the number of α such that g_α is non-singular. Then we have the theorem:

THEOREM 2.5. *$\pi_1(M)$ is 2-efficient if M is a 3-manifold.*

We shall prove the theorem by induction on $n(M, \gamma)$ after proving some lemmas:

LEMMA 2.6. *If $n(M, \gamma) = 0$, then $H_2(M; Z) \cong H_2(\pi_1(M); Z)$.*

We attach one of P^3 to M for each α, by identifying the subspace P^2 with $g_\alpha(S^2)$. This does not affect H_2. Also we have killed off π_2. By attaching n-cells ($n > 3$), we obtain a $K(\pi_1(M), 1)$. This does not affect H_2, either. Lemma 2.6 follows.

LEMMA 2.7. *If $n(M, \gamma) = 0$, and M is closed, then Theorem (2.5) is true.*

$$
\begin{aligned}
0 &= \operatorname{rank}_2 H_1(M; Z_2) - \operatorname{rank}_2 H_2(M; Z_2), && \text{by Poincaré duality,}\\
&= \operatorname{rank}_2 H_1(\pi_1(M); Z_2) - \operatorname{rank}_2 H_2(\pi_1(M); Z_2), && \text{by (2.6),}\\
&\geqslant \operatorname{def} \pi_1(M), && \text{by (1.5),}\\
&\geqslant 0, && \text{by (2.3).}
\end{aligned}
$$

Lemma 2.7 follows.

LEMMA 2.8. *If $n(M, \gamma) = 0$ and $\operatorname{Bd} M \neq \emptyset$, then Theorem (2.5) is true.*

$$
\begin{aligned}
1 - \chi(M) &= \operatorname{rank}_2 H_1(M; Z_2) - \operatorname{rank}_2 H_2(M; Z_2)\\
&= \operatorname{rank}_2 H_1(\pi_1(M); Z_2) - \operatorname{rank}_2 H_2(\pi_1(M); Z_2), && \text{by (2.6),}\\
&\geqslant \operatorname{def} \pi_1(M), && \text{by (1.5),}\\
&\geqslant 1 - \chi(M), && \text{by (2.2).}
\end{aligned}
$$

Lemma 2.8 follows.

We can now proceed with the proof by induction on $n(M, \gamma)$. If $n(M, \gamma) > 0$, there is at least one 2-sphere $g_\alpha(S^2)$ (denoted S).

LEMMA 2.9. *If S separates M, then Theorem 2.5 follows from the induction hypothesis.*

If we cut M at S and fill in the boundary 2-spheres thus created with 3-balls, we get two 3-manifolds M_1 and M_2. Let γ_1 denote the set of maps g_α (except for the one giving rise to S) whose images lie in M_1, and similarly for γ_2. Then γ_1 is a set of $\pi_1(M_1)$-generators of $\pi_2(M_1)$ and similarly for γ_2. We see this by attaching 3-balls to M, using all the maps g_α and then taking the universal cover. From a Mayer–Vietoris

sequence in the universal cover, we see that attaching 3-balls to M_1, by means of the maps in γ_1, kills off

$$H_2(\tilde{M}_1; Z) \cong \pi_2(\tilde{M}_1) \cong \pi_2(M_1),$$

where $\tilde{M}_1$ is the universal cover of M_1. Similarly, γ_2 is a set of $\pi_1(M_2)$-generators of $\pi_2(M_2)$. Now

$$n(M, \gamma) = n(M_1, \gamma_1) + n(M_2, \gamma_2) + 1.$$

Therefore

$$n(M_1, \gamma_1) < n(M, \gamma),$$

$$n(M_2, \gamma_2) < n(M, \gamma).$$

Therefore Theorem 2.5 holds for M_1 and M_2 by the induction hypothesis. Now

$$\pi_1(M) \cong \pi_1(M_1) * \pi_1(M_2).$$

Therefore $\pi_1(M)$ is 2-efficient by Lemma 1.6. Lemma 2.9 follows.

LEMMA 2.10. *If S does not separate M, then Theorem 2.5 follows from the induction hypothesis.*

If we cut M at S and fill in the boundary 2-spheres thus created, we obtain a 3-manifold M'. Let γ' be the set of maps g_α other than the one giving rise to S. Then $\pi_2(M')$ is $\pi_1(M')$-generated by the set γ'. We see this by examining the covering space of M, each sheet of which is homeomorphic to $M - S$, and in which we cross to a new sheet, whenever we cross S. In this space, we apply a similar argument to that in the proof of Lemma 2.9. Now

$$n(M, \gamma) = n(M', \gamma') + 1,$$

and so

$$n(M', \gamma') < n(M, \gamma).$$

Therefore Theorem 2.5 is true for M' by the induction hypothesis. Now

$$\pi_1(M) \cong \pi_1(M') * Z.$$

So Theorem 2.10 follows from Lemma 1.6 since Z is 2-efficient.

The proof of Theorem 2.5 is complete.

3. Closed 3-manifolds

Let M be a closed 3-manifold. Let γ be a set of $\pi_1(M)$-generators of $\pi_2(M)$ as in the projective plane theorem. The image of the Hurewicz homomorphism $\pi_2(M) \longrightarrow H_2(M; Z_2)$ is generated by the maps g_α. So, in determining the image, we can neglect those g_α which identify antipodal points, and those g_α such that $g_\alpha(S^2)$ separates M. Let $\nu(M)$ be the maximum number of times we can cut M at a 2-sphere $g_\alpha(S^2)$ for some α, such that M remains connected. Let δ be a subset of γ containing $\nu(M)$ such 2-spheres (written $g_\beta(S^2)$). We begin with the lemma:

LEMMA 3.1. *If M is closed, then*

$$\operatorname{rank}_2 \operatorname{Im}(\pi_2(M) \longrightarrow H_2(M; Z_2)) = \nu(M) = \operatorname{def} \pi_1(M).$$

Suppose M triangulated so that $g_\alpha(S^2)$ is simplicial for each α. We cut M at each $g_\beta(S^2)$, to obtain a manifold M' with boundary the union of $2\nu(M)$ 2-spheres. Any union of 3-simplexes in M' whose boundary lies entirely in $\operatorname{Bd} M'$ must cover the whole of M'. Therefore any union of 3-simplexes, in M, whose boundary consists entirely of 2-spheres $g_\beta(S^2)$, must have each $g_\beta(S^2)$ occurring twice in its boundary. Therefore there is no relation between the elements in $H_2(M; Z_2)$ represented by the 2-spheres $g_\beta(S^2)$. Therefore

$$\nu(M) \leqslant \operatorname{rank}_2 \operatorname{Im}(\pi_2(M) \longrightarrow H_2(M; Z_2)).$$

On the other hand, each 2-sphere $g_\alpha(S^2)$ is homologous to a sum of the 2-spheres $g_\beta(S^2)$ since $g_\alpha(S^2)$ separates M', or $g_\alpha(S^2) = g_\beta(S^2)$ for some β. The first equality of Lemma 3.1 follows.

We can make M into a $K(\pi_1(M), 1)$ by attaching 3-balls to M, by means of the maps g, and then adding n-cells ($n > 3$). Therefore

$$H_2(\pi_1(M); Z_2) + \operatorname{Im}(\pi_2(M) \longrightarrow H_2(M; Z_2)) \cong H_2(M; Z_2).$$

Therefore

$$\operatorname{rank}_2 H_2(\pi_1(M); Z_2) + \nu(M) = \operatorname{rank}_2 H_2(M; Z_2)$$

by the first equality of Lemma 3.1. By Theorem 2.5,

$$\begin{aligned} \operatorname{def} \pi_1(M) &= \operatorname{rank}_2 H_1(\pi_1(M); Z_2) - \operatorname{rank}_2 H_2(\pi_1(M); Z_2) \\ &= \operatorname{rank}_2 H_1(M; Z_2) - \operatorname{rank}_2 H_2(M; Z_2) + \nu(M) \\ &= \nu(M) \quad \text{by Poincaré duality.} \end{aligned}$$

Lemma 3.1 follows.

If we cut M at a 2-sphere S which does not separate M, and fill in the boundary 2-spheres thus created with 3-balls, we get a closed 3-manifold M'. Then $\pi_1(M) \cong \pi_1(M') * Z$. Therefore we have the lemma:

LEMMA 3.2. *For some closed 3-manifold N,*

$$\pi_1(M) \cong \pi_1(N) * Z * Z * \ldots * Z,$$

where there are $\nu(M)$ cyclic infinite free factors.

It is possible to prove that $\pi_1(N)$ contains no cyclic infinite free factors, but we will not concern ourselves with this point. We have the theorem:

THEOREM 3.3. *The following is a complete list of abelian groups which can be fundamental groups of closed 3-manifolds*: Z, $Z+Z+Z$, $Z+Z_2$, *and* Z_r.

This theorem was first proved by Reidemeister (5).

Since a free product is not abelian, we see from Lemma 3.2 that $\nu(M) = 0$ unless $\pi_1(M) \cong Z$. Excluding the case $\pi_1(M) \cong Z$, we see from Lemma 3.1 that $\operatorname{def}\pi_1(M) = 0$. The only abelian groups whose deficiencies are zero are those listed and also $Z+Z_r$ $(r > 2)$.

Let $\pi_1(M) \cong Z+Z_r$. Since $Z+Z_r$ is not a free product, we see from van Kampen's theorem that there is no essential 2-sphere in M. (We recall that a compact 3-manifold with only one boundary component and zero fundamental group, is a homotopy 3-ball). By Lemma 2.6,

$$H_2(M; Z) \cong H_2(\pi_1(M); Z) \cong Z_r.$$

From Lemma 2.4 we see that $r = 2$. Theorem 3.3 follows. We conclude with the theorem:

THEOREM 3.4. *If M is a closed 3-manifold, and $\pi_1(M)$ has only one non-trivial relation, then*

$$\pi_1(M) \cong Z*\ldots*Z*Z_r \text{ or } Z*\ldots*Z.$$

Let the presentation of $\pi_1(M)$ with only one relation have n generators. By Lemma 1.7, $\operatorname{def}\pi_1(M) = n-1$. By Lemma 3.1, $\operatorname{def}\pi_1(M) = \nu(M)$. By Lemma 3.2, $\pi_1(M) \cong A*Z*\ldots*Z$ with $n-1$ cyclic infinite free factors. By Grusko's Theorem [(3) 58], using the n generators of $\pi_1(M)$, we easily see that A is isomorphic to some quotient group of Z.

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SINGULAR SYSTEMS OF AN EVEN NUMBER OF PARTIAL DIFFERENTIAL EQUATIONS, OF MAXIMUM RANK

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We consider a system of n differential equations of the first order, with two independent variables x and y, and n dependent variables $z_1,\ldots,z_n$,

$$\phi_i(x, y, z_1,\ldots,z_n, p_1,\ldots,p_n, q_1,\ldots,q_n) = 0$$
$$(p_i = \partial z_i/\partial x,\ q_i = \partial z_i/\partial y,\ i = 1,\ldots,n).$$

The system is singular (**1**) if, for general values of x, y, $z_1,\ldots,z_n$, $p_1,\ldots,p_n$, $q_1,\ldots,q_n$ satisfying the equations of the system, the determinant

$$\Delta(dy, dx) \equiv \left\| \frac{\partial \phi_i}{\partial p_k}\, dy - \frac{\partial \phi_i}{\partial q_k}\, dx \right\|$$

is identically zero for all values of the ratio dy/dx. The system is of rank r ($r \geqslant 1$) if the matrix

$$M \equiv \left[\frac{\partial \phi_i}{\partial p_k}\, dy - \frac{\partial \phi_i}{\partial q_k}\, dx \right]$$

is of rank $n-r$, for general values of dy/dx, and for general values of the variables satisfying the equations of the system [see (**4**)].

Furthermore (**4**), the rank of the system is equal to the number of total differential equations which can be deduced by eliminating $p_1,\ldots,p_n$, $q_1,\ldots,q_n$ between the equations of the system and the equations

$$dz_i - p_i\, dx - q_i\, dy = 0 \quad (i = 1,\ldots,n).$$

It follows that any two singular systems which are algebraically equivalent must have the same rank.

We suppose the system analytic and of fully reduced form, i.e. solved for n of the variables x, y, $z_1,\ldots,z_n$, $p_1,\ldots,p_n$, $q_1,\ldots,q_n$. Then we can at once remark that $2r \leqslant n$. For taking $dy = 0$, $dx = 1$, we see that

$$\frac{\partial(\phi_1,\ldots,\phi_n)}{\partial(q_1,\ldots,q_n)}$$

is of rank not exceeding $n-r$ for general values of the variables satisfying the equations of the singular system. Hence we can deduce at least r distinct equations not containing $q_1,\ldots,q_n$. Similarly, we can deduce at least r equations not containing $p_1,\ldots,p_n$. Since we also suppose, as usual, that we cannot deduce, as an algebraic consequence

of the given system, an equation not containing any partial derivative, it follows that these last equations, at least $2r$ in number, are distinct: hence $2r \leqslant n$.

Now, the difficulty of constructing general existence theorems for singular systems appears to be connected with the fact that these systems are not of general form: the functions ϕ_i have to satisfy the partial differential equations which express the fact that the matrix M is of rank $n-r$ only. Thus it is of interest to determine the possible forms of various classes of singular systems. The following two cases have already been fully investigated:

(i) *Singular systems of two equations.* I have shown (**2**) that these are necessarily linear and equivalent to a single, linear total differential equation.

(ii) *Singular systems of three equations.* These are necessarily of rank 1. For, since $2r \leqslant 3$, it follows that $r = 1$. I have shown (**3**) that these systems are of three different types: namely, Type I, equivalent to a linear total differential equation, together with a partial differential equation of the first order of any form; Type II, equivalent to a single, quadratic total differential equation; Type III, obtained from the elimination of an additional 'parametric' variable, whose differential does not occur, between two linear total differential equations.

We shall now consider singular systems of an even number $2m$ of equations, of maximum rank (i.e. of rank m). Since m total differential equations can be deduced from such a system, and since a total differential equation of any form is equivalent to at least two partial differential equations, we should expect to find that the given system is entirely equivalent to the m total differential equations. We shall see that this is so and furthermore that the latter are linear.

The matrix M being, by hypothesis, of rank m, we see as above that we can deduce from the given system m distinct equations not containing $q_1,\ldots, q_{2m}$, and m more not containing $p_1,\ldots, p_{2m}$. Since we suppose that we cannot deduce a finite equation, the former may be supposed solved for m of $p_1,\ldots, p_{2m}$, say for $p_1,\ldots, p_m$. Thus let the given system be

$$\left.\begin{aligned} p_i + f_i(x, y, z_1,\ldots, z_{2m}, p_{m+1},\ldots, p_{2m}) &= 0 \quad (i = 1,\ldots, m) \\ F_j(x, y, z_1,\ldots, z_{2m}, q_1,\ldots, q_{2m}) &= 0 \quad (j = 1,\ldots, m) \end{aligned}\right\}, \tag{1}$$

the last m equations being supposed solved for m of the variables x, y, $z_1,\ldots, z_{2m}$, $q_1,\ldots, q_{2m}$.

Then either we can solve the last m equations of (1) for $q_1,\ldots, q_m$ (for general values of the variables satisfying the system) or we cannot.

In the latter event, we can eliminate $q_1,\ldots, q_m$ between these equations, obtaining at least one equation not containing $q_1,\ldots, q_m$, but containing at least one of $q_{m+1},\ldots, q_{2m}$. Let us suppose that such an equation contains q_{m+1}. Then, for general values of the variables satisfying (1) (by our hypothesis that the system is analytic and solved for $2m$ of the variables), we can solve the latter equation for q_{m+1} and write the system in the form

$$\left.\begin{aligned} p_i+f_i(x,y,z_1,\ldots,z_{2m},p_{m+1},\ldots,p_{2m}) &= 0 \quad (i=1,\ldots,m)\\ q_{m+1}+F(x,y,z_1,\ldots,z_{2m},q_{m+2},\ldots,q_{2m}) &= 0\\ G_j(x,y,z_1,\ldots,z_{2m},q_1,\ldots,q_{2m}) &= 0 \quad (j=1,\ldots,m-1)\end{aligned}\right\}, \tag{2}$$

the functions G_j not containing q_{m+1}.

But the determinant formed from the first $m+1$ rows and columns of the matrix M, for the system (2), is

$$\begin{vmatrix} dy & 0 & \cdot & \cdot & \cdot & 0 & (\partial f_1/\partial p_{m+1})\,dy\\ 0 & dy & \cdot & \cdot & \cdot & 0 & (\partial f_2/\partial p_{m+1})\,dy\\ \cdot & \cdot & \cdot & & & \cdot & \cdot\\ \cdot & \cdot & & \cdot & & \cdot & \cdot\\ \cdot & \cdot & & & \cdot & \cdot & \cdot\\ 0 & 0 & \cdot & \cdot & \cdot & dy & (\partial f_m/\partial p_{m+1})\,dy\\ 0 & 0 & \cdot & \cdot & \cdot & 0 & -dx \end{vmatrix} \equiv -dx(dy)^m.$$

This determinant is non-zero when $dxdy \neq 0$, contrary to the hypothesis that the given system is of rank m.

We thus reach a contradiction by assuming that the last m equations (1) cannot be solved for $q_1,\ldots, q_m$: and, accordingly, we suppose that the given system is written in the form

$$\left.\begin{aligned} p_i+f_i(x,y,z_1,\ldots,z_{2m},p_{m+1},\ldots,p_{2m}) &= 0\\ q_i+g_i(x,y,z_1,\ldots,z_{2m},q_{m+1},\ldots,q_{2m}) &= 0\end{aligned}\right\} \quad (i=1,\ldots,m). \tag{3}$$

Equating to zero the determinant formed from the first $m+1$ rows, and from the first m and the $(m+k)$th column, of the matrix M for the system (3), we have

$$\begin{vmatrix} dy & 0 & \cdot & \cdot & \cdot & 0 & (\partial f_1/\partial p_{m+k})\,dy\\ 0 & dy & \cdot & \cdot & \cdot & 0 & (\partial f_2/\partial p_{m+k})\,dy\\ \cdot & \cdot & \cdot & & & \cdot & \cdot\\ \cdot & \cdot & & \cdot & & \cdot & \cdot\\ \cdot & \cdot & & & \cdot & \cdot & \cdot\\ 0 & 0 & \cdot & \cdot & \cdot & dy & (\partial f_m/\partial p_{m+k})\,dy\\ -dx & 0 & \cdot & \cdot & \cdot & 0 & -(\partial g_1/\partial q_{m+k})\,dx \end{vmatrix}$$

$$\equiv dx(dy)^m\{\partial f_1/\partial p_{m+k}-\partial g_1/\partial q_{m+k}\} \equiv 0,$$

i.e., since dx and dy are arbitrary,

$$\partial f_1/\partial p_{m+k} = \partial g_1/\partial q_{m+k}.$$

Again, interchanging the first and ith rows ($i \leqslant m$), the $(m+1)$th and $(m+i)$th rows, and the first and ith columns of the matrix, and proceeding as before, we obtain

$$\partial f_i/\partial p_{m+k} = \partial g_i/\partial q_{m+k} \quad (i, k = 1,..., m).$$

But $g_1,..., g_m$ do not contain $p_{m+1},..., p_{2m}$, while $f_1,..., f_m$ do not contain $q_{m+1},..., q_{2m}$. We therefore have

$$\partial f_i/\partial p_{m+k} = \partial g_i/\partial q_{m+k} = a_{ik}(x, y, z_1,..., z_{2m}) \quad (i, k = 1,..., m).$$

Integrating, we obtain

$$\left.\begin{aligned} f_i &= \sum_{k=1}^{m} a_{ik} p_{m+k} + b_i \\ g_i &= \sum_{k=1}^{m} a_{ik} q_{m+k} + c_i \end{aligned}\right\} \quad (i = 1,..., m),$$

where all the a_{ik}, b_i, c_i are functions of x, y, $z_1,..., z_{2m}$ only. The given system is therefore equivalent to the linear system

$$\left.\begin{aligned} p_i + \sum_{k=1}^{m} a_{ik} p_{m+k} + b_i &= 0 \\ q_i + \sum_{k=1}^{m} a_{ik} q_{m+k} + c_i &= 0 \end{aligned}\right\} \quad (i = 1,..., m), \tag{4}$$

which is easily seen to be of rank m since the $(m+i)$th row of the associated matrix M is proportional to the ith row, while the leading diagonal minor of order m is equal to $(dy)^m$.

The system (4) is clearly entirely equivalent to the system of m distinct equations of Pfaff

$$dz_i + \sum_{k=1}^{m} a_{ik}\, dz_{m+k} + b_i\, dx + c_i\, dy = 0 \quad (i = 1,..., m), \tag{5}$$

and we have the result that every singular system of $2m$ equations, of rank m, is equivalent to a system of Pfaff of the form (5).

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THE GENUS OF AN ALGEBRAIC CURVE

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1. Introduction

In the classical formula

$$p = \tfrac{1}{2}(n-1)(n-2)-\tfrac{1}{2}\sum r_i(r_i-1)$$

for the genus of an irreducible plane algebraic curve C, the presence of a singular point P on C appears to reduce the genus of C by a finite sum $\frac{1}{2}\sum r_i(r_i-1)$, where the r_i are the multiplicities of C at P and at the points in the neighbourhood of P. In (**1**) Apéry showed that, if C consists of a single branch at P, then this reducing effect can be calculated without first calculating the multiplicities r_i. More recently Hironaka [(**2**) Theorem 1] and Northcott [(**6**) Theorem 4] have generalized Apéry's result to include the case in which C does not consist of a single branch at P.

By constructing a projectively normal model of an algebraic curve Hironaka [(**2**) Theorem 2] has obtained the genus formula for a curve in space of any dimensions defined over an algebraically closed ground-field. The present definition of the genus is a generalization of Hironaka's formula to (composite) curves defined over an arbitrary ground-field. Since a curve can acquire further singularities when the ground-field is extended [see Zariski (**10**) 2, examples 1 and 3], such an extension may alter the genus of a curve; thus the main result of the present paper cannot be deduced from Hironaka's result in any simple manner.

In §§ 2–4 we use the concepts and results of (**3**) and (**4**) in order to demonstrate the invariance of the genus of a curve under birational transformations. The final section is devoted to proving that, if the ground-field is algebraically closed, then the genus of every irreducible curve is non-negative.

2. The local ring of a variety at a sub-variety

Let K be an arbitrary field, and let $K[x_0,\ldots,x_m] = K[x]$ denote the polynomial ring in $m+1$ variables $x_0,\ldots, x_m$ with $m \geqslant 2$. We shall frequently be concerned with several ideals each in a different polynomial ring. When this happens, it should be understood that each ring has the same coefficient field K, and each ideal is *homogeneous*.

In particular, if we say that $\mathfrak{b}$ is a *birational transform* of $\mathfrak{a}$, then we shall mean that $\mathfrak{a}$ and $\mathfrak{b}$ are homogeneous ideals in different polynomial rings and that they are birationally equivalent in the sense of (**4**).

Let $\mathfrak{p}$ be a homogeneous prime ideal of $K[x]$ with positive dimension, and let $\mathfrak{a}$ be a homogeneous ideal of $K[x]$ which is contained in $\mathfrak{p}$. The local ring of $\mathfrak{a}$ at $\mathfrak{p}$ is denoted by $Q(\mathfrak{p}/\mathfrak{a})$; $Q(\mathfrak{p}/\mathfrak{a})$ is the residue ring of the ring of quotients of forms of the same degree $\phi(x)/\psi(x)$, where $\psi(x) \notin \mathfrak{p}$, modulo the ideal with numerators $\phi(x) \in \mathfrak{a}$. In the case when $\mathfrak{a} = \mathfrak{p}$ the local ring of $\mathfrak{p}$ at $\mathfrak{p}$ is simply the field of rational functions on $\mathfrak{p}$.

If $\lambda(x)\phi(x) \in \mathfrak{a}$ for some $\lambda(x) \notin \mathfrak{p}$, then for a suitable form $\psi(x) \notin \mathfrak{p}$ we see that

$$\frac{\phi(x)}{\psi(x)} = \lambda(x)\phi(x)/\lambda(x)\psi(x)$$

is a representative of the zero in $Q(\mathfrak{p}/\mathfrak{a})$. Thus the set of all numerators of representatives of zero in $Q(\mathfrak{p}/\mathfrak{a})$ is the set of forms of the $(K[x]-\mathfrak{p})$-component of $\mathfrak{a}$ [see (**5**) 17]. We shall regard the zero ideal of $Q(\mathfrak{p}/\mathfrak{a})$ as the extension of $\mathfrak{a}$ to $Q(\mathfrak{p}/\mathfrak{a})$, and the $(K[x]-\mathfrak{p})$-component of $\mathfrak{a}$ as the contraction to $K[x]$ of the zero ideal of $Q(\mathfrak{p}/\mathfrak{a})$.

In a similar manner, if $\mathfrak{b}$ is a homogeneous ideal of $K[x]$ such that $\mathfrak{p} \supseteq \mathfrak{b} \supseteq \mathfrak{a}$, then we define the *extension* $E(\mathfrak{b})$ of $\mathfrak{b}$ to $Q(\mathfrak{p}/\mathfrak{a})$ as the ideal of $Q(\mathfrak{p}/\mathfrak{a})$ for which each element has at least one representative $\phi(x)/\psi(x)$ such that $\phi(x) \in \mathfrak{b}$. If $\bar{\mathfrak{b}}$ is a proper ideal of $Q(\mathfrak{p}/\mathfrak{a})$, we define the *contraction* $C(\bar{\mathfrak{b}})$ of $\bar{\mathfrak{b}}$ to $K[x]$ as the homogeneous ideal whose forms are the numerators of representatives $\phi(x)/\psi(x)$ of elements of $\bar{\mathfrak{b}}$. Clearly

$$E(C(\bar{\mathfrak{b}})) = \bar{\mathfrak{b}} \quad \text{and} \quad C(E(\mathfrak{b})) \supseteq \mathfrak{b}.$$

In fact $C(E(\mathfrak{b}))$ is the $(K[x]-\mathfrak{p})$-component of $\mathfrak{b}$, so that $C(E(\mathfrak{b})) = \mathfrak{b}$ if and only if every prime belonging to $\mathfrak{b}$ is contained in $\mathfrak{p}$ [see (**5**) 17, Proposition, 7]. Consequently there is a one-one correspondence between the proper ideals of $Q(\mathfrak{p}/\mathfrak{a})$ and the homogeneous ideals of $K[x]$ which contain $\mathfrak{a}$ and whose primes are all contained in $\mathfrak{p}$.

This one-one correspondence leads to the following results which we shall need later.

(*a*) If $\mathfrak{q}$ is a homogeneous $\mathfrak{p}$-primary ideal of $K[x]$ which contains $\mathfrak{a}$, then $C(E(\mathfrak{q})) = \mathfrak{q}$ and $E(\mathfrak{q})$ is primary for the maximal ideal $E(\mathfrak{p})$ of $Q(\mathfrak{p}/\mathfrak{a})$.

(*b*) If $\mathfrak{q}$ is as in (*a*), then the length of a composition series of $\mathfrak{p}$-primary homogeneous ideals from $\mathfrak{q}$ to $\mathfrak{p}$ is equal to the length of $E(\mathfrak{q})$ as an ideal of $Q(\mathfrak{p}/\mathfrak{a})$.

(*c*) If $\mathfrak{b}$ is a homogeneous ideal of $K[x]$ such that $\mathfrak{p}$ is a minimal prime ideal of $\mathfrak{b}+\mathfrak{a}$ and if $\mathfrak{q}$ denotes the $\mathfrak{p}$-primary component of $\mathfrak{b}+\mathfrak{a}$, then

$$E(\mathfrak{b}+\mathfrak{a}) = E(\mathfrak{q}) \quad \text{and} \quad E(\mathfrak{b}^n+\mathfrak{a}) = \{E(\mathfrak{q})\}^n$$

for all $n \geqslant 1$. Thus, by (*b*), the length of a composition sequence of homogeneous $\mathfrak{p}$-primary ideals from the $\mathfrak{p}$-primary component of $\mathfrak{b}^n+\mathfrak{a}$ to $\mathfrak{p}$ is equal to the length of $\{E(\mathfrak{q})\}^n$ as an ideal of $Q(\mathfrak{p}/\mathfrak{a})$.

In the present paper we shall usually be concerned with the situation described above for the particular case in which $\mathfrak{p}$ has dimension 1 and $\mathfrak{a}$ is the intersection of a finite number of two-dimensional (homogeneous) prime ideals. Thus, in geometric language, we shall be considering the local ring of a (composite) curve at a point; $Q(\mathfrak{p}/\mathfrak{a})$ is then a one-dimensional geometric local ring whose maximal ideal does not consist entirely of zero-divisors. Hence the zero ideal of the completion of $Q(\mathfrak{p}/\mathfrak{a})$ is the intersection of prime ideals, so $Q(\mathfrak{p}/\mathfrak{a})$ has a finite defect [see (**3**) Theorem 4] which we shall denote by $\Delta[Q(\mathfrak{p}/\mathfrak{a})]$.

3. The genus of a curve

Let $\mathfrak{a}$ be an ideal of $K[x]$ which is the intersection of a finite number of two-dimensional homogeneous prime ideals, and let $H(\alpha;\mathfrak{a})$ be the Hilbert function of $\mathfrak{a}$, i.e. $H(\alpha;\mathfrak{a})$ is the number of forms of degree α in $K[x]$ which are linearly independent modulo $\mathfrak{a}$. Since $\mathfrak{a}$ is two-dimensional, for α sufficiently large $H(\alpha;\mathfrak{a})$ has a polynomial expression $\chi(\alpha;\mathfrak{a})$, where $\chi(\alpha;\mathfrak{a})$ is of degree 1 in α. If we write

$$\chi(\alpha;\mathfrak{a}) = A\alpha+(1-B),$$

then A is usually called the *order* of $\mathfrak{a}$, $\mathrm{ord}(\mathfrak{a})$. We shall call the integer B the *apparent genus* of $\mathfrak{a}$ and denote it by $\bar{p}(\mathfrak{a})$.

We define the *genus* of $\mathfrak{a}$ as the integer

$$p(\mathfrak{a}) = \bar{p}(\mathfrak{a}) - \sum \mathrm{ord}(\mathfrak{p}_i)\,\Delta[Q(\mathfrak{p}_i/\mathfrak{a})], \tag{1}$$

where the sum is taken over all one-dimensional homogeneous prime ideals $\mathfrak{p}_i$ which contain $\mathfrak{a}$, and $\mathrm{ord}(\mathfrak{p}_i)$ is the positive integer $\chi(\alpha;\mathfrak{p}_i)$.

We now show that the sum in (1) is effectively finite. Let

$$\mathfrak{a} = \bar{\mathfrak{p}}_1 \cap \ldots \cap \bar{\mathfrak{p}}_s$$

be a normal decomposition of $\mathfrak{a}$, so that the $\bar{\mathfrak{p}}_j$ are two-dimensional prime ideals. Only a finite number of one-dimensional homogeneous prime ideals $\mathfrak{p}_i$ can contain more than one $\bar{\mathfrak{p}}_j$, namely the one-dimensional primes belonging to the $\frac{1}{2}s(s-1)$ ideals $\bar{\mathfrak{p}}_j+\bar{\mathfrak{p}}_k$ $(j \neq k)$. But, if $\mathfrak{p}_i$ contains $\bar{\mathfrak{p}}_j$ only, $Q(\mathfrak{p}_i/\mathfrak{a}) = Q(\mathfrak{p}_i/\bar{\mathfrak{p}}_j)$ and $Q(\mathfrak{p}_i/\bar{\mathfrak{p}}_j)$ is a regular one-dimensional local ring for all but a finite number of ideals $\mathfrak{p}_i$ [see (**10**) corollary

to Theorem 11]. Moreover, if $Q(\mathfrak{p}_i/\mathfrak{a})$ is a regular local ring, it is integrally closed and $\Delta[Q(\mathfrak{p}_i/\mathfrak{a})] = 0$. Therefore $\Delta[Q(\mathfrak{p}_i/\mathfrak{a})] \neq 0$ for only a finite number of one-dimensional prime ideals $\mathfrak{p}_i$.

If $Q(\mathfrak{p}_i/\mathfrak{a})$ is a regular local ring for all $\mathfrak{p}_i \supset \mathfrak{a}$, we shall say that $\mathfrak{a}$ *represents a non-singular curve*; thus if $\mathfrak{a}$ represents a non-singular curve we see from (1) that the genus of $\mathfrak{a}$ is equal to the apparent genus of $\mathfrak{a}$. Conversely, if the genus of $\mathfrak{a}$ is equal to the apparent genus of $\mathfrak{a}$, then $\Delta[Q(\mathfrak{p}_i/\mathfrak{a})] = 0$ for all one-dimensional homogeneous ideals $\mathfrak{p}_i \supset \mathfrak{a}$. And, since the maximal ideal of $Q(\mathfrak{p}_i/\mathfrak{a})$ does not consist entirely of zero divisors, this implies that the reduction number of $Q(\mathfrak{p}_i/\mathfrak{a})$ is zero [(3) Theorem 4]. Therefore $Q(\mathfrak{p}_i/\mathfrak{a})$ is a regular local ring for all $\mathfrak{p}_i$ [(7) Theorem 2]. Thus we have proved the theorem:

THEOREM 1. *The ideal* $\mathfrak{a}$ *represents a non-singular curve if and only if its genus is equal to its apparent genus.*

4. The invariance of the genus

Using the definition of birational transformations given in (4) we devote the present section to proving that, if the ideal $\mathfrak{b}$ is a birational transform of $\mathfrak{a}$, then $p(\mathfrak{b}) = p(\mathfrak{a})$. We do this in two stages. Firstly we show that, if $\mathfrak{a}$ and $\mathfrak{b}$ represent non-singular curves, then $\bar{p}(\mathfrak{a}) = \bar{p}(\mathfrak{b})$; and secondly we prove that $\mathfrak{a}$ has a birational transform $\mathfrak{a}'$ which represents a non-singular curve such that $p(\mathfrak{a}) = \bar{p}(\mathfrak{a}')$.

Let $\mathfrak{a}$ be the intersection of a finite number s of two-dimensional homogeneous prime ideals, and let $\mathfrak{b}$ be a birational transform of $\mathfrak{a}$. Then $\mathfrak{b}$ is also the intersection of s two-dimensional prime ideals [(4) Corollary 1 of Theorem 5]. Let

$$y_j = f_j(x) \quad (j = 0,\ldots,n), \qquad x_i = g_i(y) \quad (i = 0,\ldots,m)$$

be the equations of the induced rational transformations between $\mathfrak{a}$ and $\mathfrak{b}$, where $\mathfrak{a}$ is an ideal of $K[x_0,\ldots,x_m]$ and $\mathfrak{b}$ is an ideal of $K[y_0,\ldots,y_n]$. Using the notation of (4) we see from equation (5) of § 3 of that paper that

$$\chi(\alpha+\mu\beta;\mathfrak{a}) - \chi(\alpha,\beta;\mathfrak{a},\mathfrak{f}) = \chi(\beta+\nu\alpha;\mathfrak{b}) - \chi(\beta,\alpha;\mathfrak{b},\mathfrak{g}), \tag{2}$$

where μ and ν are the degrees of the forms $f_j(x)$ and $g_i(y)$ respectively, and $\mathfrak{f}$ and $\mathfrak{g}$ denote the ideals $(f_0(x),\ldots,f_n(x))$ of $K[x]$ and $(g_0(y),\ldots,g_m(y))$ of $K[y]$ respectively. The polynomials $\chi(\alpha+\mu\beta;\mathfrak{a})$, $\chi(\beta+\nu\alpha;\mathfrak{b})$ can be obtained directly from the Hilbert polynomials of $\mathfrak{a}$ and $\mathfrak{b}$. We wish also to calculate $\chi(\alpha,\beta;\mathfrak{a},\mathfrak{f})$ and $\chi(\beta,\alpha;\mathfrak{b},\mathfrak{g})$. It will, of course, suffice to consider $\chi(\alpha,\beta;\mathfrak{a},\mathfrak{f})$, so we require a polynomial expression for the Hilbert function $H(\alpha+\mu\beta;\mathfrak{a}+\mathfrak{f}^\beta)$.

Since $\mathfrak{a}\colon\mathfrak{f} = \mathfrak{a}$ [(4) Theorem 1], $\mathfrak{f}$ is contained in no prime ideal of $\mathfrak{a}$.

Therefore $\mathfrak{a}+\mathfrak{f}$ has dimension 0 or 1. Moreover, the radicals of $\mathfrak{a}+\mathfrak{f}^\beta$ are the same for all $\beta \geqslant 1$; hence either $\mathfrak{a}+\mathfrak{f}^\beta$ is primary for

$$\mathfrak{X} = (x_0, \ldots, x_m)$$

or $\mathfrak{a}+\mathfrak{f}^\beta$ has a set of minimal prime ideals $\mathfrak{p}_1, \ldots, \mathfrak{p}_r$ which are the same for all $\beta \geqslant 1$. In the first case $H(\alpha+\mu\beta; \mathfrak{a}+\mathfrak{f}^\beta) = 0$ for each β when α is large enough. Whence we deduce that the polynomial $\chi(\alpha, \beta; \mathfrak{a}, \mathfrak{f})$ is zero at infinitely many values of α for each large β, i.e. $\chi(\alpha, \beta; \mathfrak{a}, \mathfrak{f}) = 0$. In the second case we fix β and note that, for all large α, $H(\alpha+\mu\beta; \mathfrak{a}+\mathfrak{f}^\beta)$ has a polynomial expression

$$\chi(\alpha+\mu\beta; \mathfrak{a}+\mathfrak{f}^\beta) = \operatorname{ord}(\mathfrak{a}+\mathfrak{f}^\beta) = \sum_{i=1}^{r} \operatorname{ord}(\mathfrak{p}_i)\mathscr{L}\{\mathfrak{q}_i(\beta)\},$$

where $\mathscr{L}\{\mathfrak{q}_i(\beta)\}$ is the length of a composition series of $\mathfrak{p}_i$-primary homogeneous ideals from the $\mathfrak{p}_i$-primary component, $\mathfrak{q}_i(\beta)$, of $\mathfrak{a}+\mathfrak{f}^\beta$ to $\mathfrak{p}_i$ [see van der Waerden (**8**)]. However, if $\bar{\mathfrak{q}}_i$ is the extension of $\mathfrak{a}+\mathfrak{f}$ to the local ring $Q(\mathfrak{p}_i/\mathfrak{a})$, then, by (*c*) of § 2, $\mathscr{L}\{\mathfrak{q}_i(\beta)\}$ is equal to the length of $\bar{\mathfrak{q}}_i^\beta$ in $Q(\mathfrak{p}_i/\mathfrak{a})$. Therefore, if β is sufficiently large,

$$\mathscr{L}\{\mathfrak{q}_i(\beta)\} = \mathscr{L}(\bar{\mathfrak{q}}_i^\beta) = e_i\beta - \rho_i,$$

where e_i is the multiplicity and ρ_i is the reduction number of $\bar{\mathfrak{q}}_i$. Consequently the polynomial

$$\chi(\alpha, \beta; \mathfrak{a}, \mathfrak{f}) - \sum_{i=1}^{r} \operatorname{ord}(\mathfrak{p}_i)(e_i\beta - \rho_i)$$

is zero at infinitely many values of α for each large β. Hence

$$\chi(\alpha, \beta; \mathfrak{a}, \mathfrak{f}) = \sum_{i=1}^{r} \operatorname{ord}(\mathfrak{p}_i)(e_i\beta - \rho_i). \tag{3}$$

When $\mathfrak{p} \supseteq \mathfrak{q} \supseteq \mathfrak{a}$ and $\mathfrak{q}$ is a $\mathfrak{p}$-primary homogeneous ideal, we shall refer to the multiplicity and reduction number of the extension of $\mathfrak{q}$ in $Q(\mathfrak{p}/\mathfrak{a})$ as the multiplicity and reduction number of $\mathfrak{q}$ itself. Thus the integers e_i, ρ_i which appear in equation (3) will be referred to as the *multiplicity* and *reduction number* of the $\mathfrak{p}_i$-primary component of the basis [(**4**) § 4] of the rational transformation from $\mathfrak{a}$ to $\mathfrak{b}$.

Similarly, we obtain a relation

$$\chi(\beta, \alpha; \mathfrak{b}, \mathfrak{g}) = \sum_{i=1}^{r'} \operatorname{ord}(\mathfrak{p}_i')(e_i'\alpha - \rho_i'), \tag{4}$$

where $\mathfrak{p}_1', \ldots, \mathfrak{p}_{r'}'$ are the one-dimensional primes of $\mathfrak{b}+\mathfrak{g}$ and e_i', ρ_i' are the multiplicity and reduction number of the $\mathfrak{p}_i'$-primary component of the basis of the rational transformation from $\mathfrak{b}$ to $\mathfrak{a}$ ($i = 1, \ldots, r'$).

Substituting (3) and (4) into equation (2) and equating coefficients we have

$$\operatorname{ord}(\mathfrak{a}) = \nu \operatorname{ord}(\mathfrak{b}) - \sum_{i=1}^{r'} \operatorname{ord}(\mathfrak{p}_i').e_i',$$

$$\operatorname{ord}(\mathfrak{b}) = \mu \operatorname{ord}(\mathfrak{a}) - \sum_{i=1}^{r} \operatorname{ord}(\mathfrak{p}_i).e_i,$$

$$\bar{p}(\mathfrak{a}) - \sum_{i=1}^{r} \operatorname{ord}(\mathfrak{p}_i).\rho_i = \bar{p}(\mathfrak{b}) - \sum_{i=1}^{r'} \operatorname{ord}(\mathfrak{p}_i').\rho_i'. \tag{5}$$

It should be noted that, if the ideal $\mathfrak{a}+\mathfrak{f}$ of $K[x]$ or the ideal $\mathfrak{b}+\mathfrak{g}$ of $K[y]$ have no one-dimensional components, then the corresponding sums in the above equations should be omitted. The first two equations above have obvious geometric interpretations: namely, if a curve U is transformed birationally into a curve V by means of primals of order μ, then the order of V is equal to μ times the order of U less the sum of the multiplicities of the base points of the transformation. Equation (5) is of a rather different character and we shall see that it implies that the genus is a birational invariant. We prove the theorem:

THEOREM 2. *If* $\mathfrak{b}$ *is a birational transform of* $\mathfrak{a}$, *and if* $\mathfrak{a}$ *and* $\mathfrak{b}$ *represent non-singular curves, then the apparent genus of* $\mathfrak{a}$ *is equal to the apparent genus of* $\mathfrak{b}$.

Proof. If $\mathfrak{p}$ is a one-dimensional homogeneous prime ideal containing $\mathfrak{a}$, then $Q(\mathfrak{p}/\mathfrak{a})$ is a regular one-dimensional local ring since $\mathfrak{a}$ represents a non-singular curve. Hence $Q(\mathfrak{p}/\mathfrak{a})$ is integrally closed and $\Delta[Q(\mathfrak{p}/\mathfrak{a})] = 0$. Therefore every ideal of $Q(\mathfrak{p}/\mathfrak{a})$ which is primary for the maximal ideal has a zero reduction number [(3) Theorem 4]. Thus the left-hand side of equation (5) for the birational transformation between $\mathfrak{a}$ and $\mathfrak{b}$ is simply $\bar{p}(\mathfrak{a})$. Similarly the right-hand side is equal to $\bar{p}(\mathfrak{b})$, and $\bar{p}(\mathfrak{a}) = \bar{p}(\mathfrak{b})$. We have also the theorem:

THEOREM 3. *The ideal* $\mathfrak{a}$ *has a birational transform* $\mathfrak{a}'$ *such that* $\mathfrak{a}'$ *represents a non-singular curve and the apparent genus of* $\mathfrak{a}'$ *is equal to the genus of* $\mathfrak{a}$.

Proof. If $\mathfrak{a}$ represents a non-singular curve, we can take $\mathfrak{a}' = \mathfrak{a}$, and the result follows from Theorem 1.

Suppose that $\mathfrak{a}$ does not represent a non-singular curve. Let $\mathfrak{p}_1,\ldots,\mathfrak{p}_r$ be the only one-dimensional homogeneous prime ideals containing $\mathfrak{a}$ for which $\Delta[Q(\mathfrak{p}_i/\mathfrak{a})] \neq 0$. Let $\bar{\mathfrak{q}}_i$ be an ideal of $Q(\mathfrak{p}_i/\mathfrak{a})$ which is primary for the maximal ideal of $Q(\mathfrak{p}_i/\mathfrak{a})$ such that the reduction number of $\bar{\mathfrak{q}}_i$ is equal to $\Delta[Q(\mathfrak{p}_i/\mathfrak{a})]$ $(i = 1,\ldots,r)$: for example we may choose $\bar{\mathfrak{q}}_i$ to

be the conductor to $Q(\mathfrak{p}_i/\mathfrak{a})$ from its integral closure in the full ring of quotients [see (3) Theorem 4]. Let $\mathfrak{q}_i$ be the contraction of $\bar{\mathfrak{q}}_i$ to $K[x]$.

We apply a monoidal transformation of degree μ to $\mathfrak{a}$ with $\mathfrak{q}_1 \cap \ldots \cap \mathfrak{q}_r$ as basis, and so obtain a birational transform $\mathfrak{a}'$ of $\mathfrak{a}$ [see (4) § 4]. Since the reduction number of $\mathfrak{q}_i$ is $\Delta[Q(\mathfrak{p}_i/\mathfrak{a})]$, we see from the concluding remark of (3) that equation (5) applied to the monoidal transformation between $\mathfrak{a}$ and $\mathfrak{a}'$ yields

$$\bar{p}(\mathfrak{a}) - \sum_{i=1}^{r} \operatorname{ord}(\mathfrak{p}_i).\Delta[Q(\mathfrak{p}_i/\mathfrak{a})] = \bar{p}(\mathfrak{a}'),$$

i.e. the genus of $\mathfrak{a}$ is equal to the apparent genus of $\mathfrak{a}'$. We must now prove that $\mathfrak{a}'$ represents a non-singular curve.

Suppose that $\mathfrak{a}'$ does not represent a non-singular curve, so that $p(\mathfrak{a}') < \bar{p}(\mathfrak{a}')$. We can now apply a monoidal transformation as above to $\mathfrak{a}'$ and construct an ideal $\mathfrak{a}''$ such that $p(\mathfrak{a}') = \bar{p}(\mathfrak{a}'')$. However $\mathfrak{a}''$ is a birational transform of $\mathfrak{a}$ and applying equation (5) we have

$$p(\mathfrak{a}) \leqslant \bar{p}(\mathfrak{a}'').$$

We have therefore proved that

$$p(\mathfrak{a}') < \bar{p}(\mathfrak{a}') = p(\mathfrak{a}) \leqslant \bar{p}(\mathfrak{a}'') = p(\mathfrak{a}'),$$

i.e. $p(\mathfrak{a}') < p(\mathfrak{a}')$. Consequently $\mathfrak{a}'$ must represent a non-singular curve.

Theorem 2 and 3 together establish the theorem.

THEOREM 4. *If $\mathfrak{b}$ is a birational transform of $\mathfrak{a}$, then the genus of $\mathfrak{b}$ is equal to the genus of $\mathfrak{a}$.*

Proof. Construct, in accordance with Theorem 3, ideals $\mathfrak{a}'$ and $\mathfrak{b}'$ such that $\mathfrak{a}'$, $\mathfrak{b}'$ represent non-singular curves, and $\mathfrak{a}$, $\mathfrak{a}'$ and $\mathfrak{b}$, $\mathfrak{b}'$ are birationally equivalent. We then have $\bar{p}(\mathfrak{a}') = p(\mathfrak{a})$ and $\bar{p}(\mathfrak{b}') = p(\mathfrak{b})$.

However, $\mathfrak{a}'$ and $\mathfrak{b}'$ must also be birationally equivalent, so that

$$\bar{p}(\mathfrak{a}') = \bar{p}(\mathfrak{b}')$$

by Theorem 2. Hence $p(\mathfrak{a}) = p(\mathfrak{b})$.

5. The genus of an irreducible curve

Our final theorem shows that, if the ground-field K is algebraically closed, then the genus of a two-dimensional prime ideal of $K[x]$ is non-negative:

THEOREM 5. *Let $\mathfrak{a}$ be a two-dimensional homogeneous prime ideal of $K[x]$, and let K' be the field of functions on $\mathfrak{a}$. If K is algebraically closed in K', then the genus of $\mathfrak{a}$ is non-negative.*

Proof. We can find a birational transform $\mathfrak{a}'$ of $\mathfrak{a}$ such that

$p(\mathfrak{a}) = \bar{p}(\mathfrak{a}')$ (Theorem 3). The field of functions on $\mathfrak{a}'$ is then K' [(**4**) Corollary 1 of Theorem 4]. Consequently it suffices to prove that the apparent genus of $\mathfrak{a}$ is non-negative.

Let $\mathfrak{X}$ denote the ideal $(x_0,\ldots,x_m)$ of $K[x]$, and let ϕ be a form of $K[x]$ of arbitrary degree α such that $\phi \notin \mathfrak{a}$. Then the ideal $\mathfrak{a}+(\phi)$ can be written in the form $\mathfrak{b} \cap \mathfrak{q}$, where $\mathfrak{X}$ is not a prime ideal belonging to the homogeneous ideal $\mathfrak{b}$, and $\mathfrak{q}$ is either $K[x]$ or an $\mathfrak{X}$-primary homogeneous ideal [(**9**) Theorem 2].

We first prove that every form θ of degree α in $\mathfrak{b}$ is also a form of $\mathfrak{a}+(\phi)$. Since $\mathfrak{q}$ is either $\mathfrak{X}$-primary or equal to $K[x]$, we can find an integer μ such that $\mathfrak{q} \supseteq \mathfrak{X}^\mu$. Let ω be any power product of degree μ in $K[x]$; then $\omega\theta \in \mathfrak{q}$ and $\theta \in \mathfrak{b}$. Therefore

$$\omega\theta \in \mathfrak{b} \cap \mathfrak{q} = \mathfrak{a}+(\phi).$$

If $\omega_1,\ldots,\omega_s$ are all the power products of degree μ, we then have

$$\omega_i\theta - \sum_{j=1}^{s} a_{ij}\omega_j\phi \in \mathfrak{a} \quad (i = 1,\ldots,s)$$

for suitable coefficients $a_{ij} \in K$. Denoting by

$$\theta^s + b_1\theta^{s-1}\phi + \ldots + b_s\phi^s$$

the expansion of the determinant $\det(\delta_{ij}\theta - a_{ij}\phi)$ we have

$$b_i \in K \quad (i = 1,\ldots,s),$$

$$\omega_i(\theta^s + b_1\theta^{s-1}\phi + \ldots + b_s\phi^s) \in \mathfrak{a} \quad (i = 1,\ldots,s),$$

so that $$\mathfrak{X}^\mu(\theta^s + b_1\theta^{s-1}\phi + \ldots + b_s\phi^s) \subseteq \mathfrak{a}.$$

But $\mathfrak{a}$ ($\neq \mathfrak{X}$) is prime; therefore

$$\theta^s + b_1\theta^{s-1}\phi + \ldots + b_s\phi^s \in \mathfrak{a}. \tag{6}$$

Since $\phi \notin \mathfrak{a}$, the quotient θ/ϕ is a representative of an element of the field of functions K' on $\mathfrak{a}$. Equation (**6**) therefore shows that θ/ϕ is a representative of an element of the algebraic closure of K in K'. But K is algebraically closed in K', so that $\theta - \lambda\phi \in \mathfrak{a}$ for some λ in K, i.e. $\theta \in \mathfrak{a}+(\phi)$.

Denoting by $\mathfrak{r}(\alpha)$ the K-module of forms of degree α in $K[x]$, we have proved that

$$\mathfrak{r}(\alpha) \cap \mathfrak{b} = \mathfrak{r}(\alpha) \cap [\mathfrak{a}+(\phi)]. \tag{7}$$

Since $\mathfrak{q}$ is $\mathfrak{X}$-primary and $\mathfrak{X}$ is not a prime belonging to $\mathfrak{b}$, we can find a form ψ of $\mathfrak{q}$ which is contained in none of the prime ideals of $\mathfrak{b}$. Moreover the degree β of ψ can be chosen arbitrarily large. Hence

$$\{\mathfrak{a}+(\phi)\}\colon(\psi) = (\mathfrak{b} \cap \mathfrak{q})\colon(\psi) = \mathfrak{b}\colon(\psi) = \mathfrak{b}.$$

Combining this result with (7) we deduce

$$\mathfrak{r}(\alpha) \cap [\{\mathfrak{a}+(\phi)\}\colon (\psi)] = \mathfrak{r}(\alpha) \cap [\mathfrak{a}+(\phi)]. \tag{8}$$

The theorem will now follow from relations between the lengths of various K-modules. Since

$$\mathfrak{a}+(\phi)+(\psi) \supseteq \mathfrak{a}+(\phi) \supseteq \mathfrak{a},$$

we have

$$\begin{aligned}\mathscr{L}_K\{\mathfrak{r}(\alpha+\beta)/\mathfrak{r}(\alpha+\beta) \cap \mathfrak{a}\} &= \mathscr{L}_K\{\mathfrak{r}(\alpha+\beta)/\mathfrak{r}(\alpha+\beta) \cap [\mathfrak{a}+(\phi)+(\psi)]\}+\\ &+\mathscr{L}_K\{\mathfrak{r}(\sigma+\beta) \cap [\mathfrak{a}+(\phi)+(\psi)]/\mathfrak{r}(\alpha+\beta) \cap [\mathfrak{a}+(\phi)]\}+\\ &+\mathscr{L}_K\{\mathfrak{r}(\alpha+\beta) \cap [\mathfrak{a}+(\phi)]/\mathfrak{r}(\alpha+\beta) \cap \mathfrak{a}\}.\end{aligned}$$

But

$$\begin{aligned}&\mathscr{L}_K\{\mathfrak{r}(\alpha+\beta) \cap [\mathfrak{a}+(\phi)+(\psi)]/\mathfrak{r}(\alpha+\beta) \cap [\mathfrak{a}+(\phi)]\}\\ &\quad= \mathscr{L}_K\{\mathfrak{r}(\alpha+\beta) \cap (\psi)/\mathfrak{r}(\alpha+\beta) \cap (\psi) \cap [\mathfrak{a}+(\phi)]\}\\ &\quad= \mathscr{L}_K\{\mathfrak{r}(\alpha)/\mathfrak{r}(\alpha) \cap [\mathfrak{a}+(\phi)\colon(\psi)]\}, \quad \text{since } \psi \text{ has degree } \beta,\\ &\quad= \mathscr{L}_K\{\mathfrak{r}(\alpha)/\mathfrak{r}(\alpha) \cap [\mathfrak{a}+(\phi)]\}, \quad \text{from equation (8)},\\ &\quad= \mathscr{L}_K\{\mathfrak{r}(\alpha)/\mathfrak{r}(\alpha) \cap \mathfrak{a}\} - \mathscr{L}_K\{\mathfrak{r}(\alpha) \cap [\mathfrak{a}+(\phi)]/\mathfrak{r}(\alpha) \cap \mathfrak{a}\}.\end{aligned}$$

However, ϕ has degree α, and so

$$\mathscr{L}_K\{\mathfrak{r}(\alpha) \cap [\mathfrak{a}+(\phi)]/\mathfrak{r}(\alpha) \cap \mathfrak{a}\} = 1.$$

Similarly,

$$\begin{aligned}\mathscr{L}_K\{\mathfrak{r}(\alpha+\beta) \cap [\mathfrak{a}+(\phi)]/\mathfrak{r}(\alpha+\beta) \cap \mathfrak{a}\} &= \mathscr{L}_K\{\mathfrak{r}(\alpha+\beta) \cap (\phi)/\mathfrak{r}(\alpha+\beta) \cap (\phi) \cap \mathfrak{a}\}\\ &= \mathscr{L}_K\{\mathfrak{r}(\beta)/\mathfrak{r}(\beta) \cap [\mathfrak{a}\colon(\phi)]\},\end{aligned}$$

since ϕ has degree α. However, $\mathfrak{a}$ is prime and $\phi \notin \mathfrak{a}$; therefore $\mathfrak{a}\colon(\phi) = \mathfrak{a}$. Thus we have proved that

$$\begin{aligned}\mathscr{L}_K\{\mathfrak{r}(\alpha+\beta)/\mathfrak{r}(\alpha+\beta) \cap \mathfrak{a}\} &= \mathscr{L}_K\{\mathfrak{r}(\alpha+\beta)/\mathfrak{r}(\alpha+\beta) \cap [\mathfrak{a}+(\phi)+(\psi)]\}\\ &+\mathscr{L}_K\{\mathfrak{r}(\alpha)/\mathfrak{r}(\alpha) \cap \mathfrak{a}\}+\mathscr{L}_K\{\mathfrak{r}(\beta)/\mathfrak{r}(\beta) \cap \mathfrak{a}\}-1. \quad (9)\end{aligned}$$

Since $\mathfrak{a}$ has dimension 2, we can find an integer α_0 such that

$$\mathscr{L}_K\{\mathfrak{r}(\alpha)/\mathfrak{r}(\alpha) \cap \mathfrak{a}\} = H(\alpha; \mathfrak{a})$$

can be represented by the polynomial expression

$$\alpha.\mathrm{ord}(\mathfrak{a}) - \bar{p}(\mathfrak{a}) + 1, \quad \text{for } \alpha \geqslant \alpha_0.$$

The degrees α, β of ϕ, ψ can be chosen arbitrarily large. Thus, if we choose $\alpha, \beta \geqslant \alpha_0$, equation (9) gives

$$(\alpha+\beta)\operatorname{ord}(\mathfrak{a})-\bar{p}(\mathfrak{a})+1 = H(\alpha+\beta;\ \mathfrak{a}+(\phi)+(\psi))+ \\ +\alpha\operatorname{ord}(\mathfrak{a})-\bar{p}(\mathfrak{a})+1+\beta\operatorname{ord}(\mathfrak{a})-\bar{p}(\mathfrak{a})+1-1,$$

i.e.
$$\bar{p}(\mathfrak{a}) = H(\alpha+\beta;\ \mathfrak{a}+(\phi)+(\psi)) \geqslant 0.$$

Therefore the apparent genus of $\mathfrak{a}$ is non-negative.

If K is not algebraically closed, then one would expect to be able to find a two-dimensional prime ideal $\mathfrak{a}$ of $K[x_0,\ldots,x_m]$, for m sufficiently large, such that the genus of $\mathfrak{a}$ is negative.

For example, if K is the field of real numbers, then the ideal

$$\mathfrak{a} = (x_0^2+x_1^2, x_2^2+x_3^2, x_0x_2+x_1x_3, x_1x_2-x_0x_3)$$

of $K[x_0, x_1, x_2, x_3]$ is prime and it represents a non-singular curve (namely, a pair of non-intersecting conjugate lines). $H(\alpha; \mathfrak{a}) = 2\alpha+2$ for $\alpha \geqslant 1$, and so the genus (and apparent genus) of $\mathfrak{a}$ is -1.

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ON THE NATURE OF THE SPECTRUM IN PROBLEMS OF RELATIVISTIC QUANTUM MECHANICS

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1. In relativistic quantum mechanics, the problem of a particle in a field of potential V which is a function of the radial coordinate r only depends on the solution of a pair of simultaneous equations of the form

$$(W-V+mc^2)\psi_1(r) = c\{\psi_2'(r)-l\psi_2(r)/r\}, \tag{1.1}$$

$$(W-V-mc^2)\psi_2(r) = -c\{\psi_1'(r)+l\psi_1(r)/r\}. \tag{1.2}$$

Here l is a positive integer, W is the energy and is the eigenvalue parameter, m is the mass of the particle, and c is the velocity of light. The units have been chosen so that the constant $\hbar$ of quantum theory is unity. The problem to be considered is that of the nature of the spectrum when the function $V = V(r)$ satisfies given conditions. By the spectrum I mean the set of values of W which contribute to the formula for the expansion of an arbitrary function in terms of the solutions of (1.1), (1.2).

A similar problem was considered by Plesset (**1**), but in his paper the spectrum is regarded as being continuous or discrete according to the asymptotic character, oscillatory or otherwise, of the solutions of the differential equations. As will be seen, the two different points of view lead to some differences in the conclusions to be drawn from the analysis. My analysis is also more general than Plesset's in that his function $V(r)$ is a polynomial in r or in $1/r$, whereas mine can be any sufficiently well-behaved function.

The conclusions reached by Plesset are as follows. If $V(r)$ is a polynomial in r, there is a continuous spectrum over the whole range $(-\infty, \infty)$. If it is a polynomial in $1/r$ of degree greater than 1, the spectrum is again continuous, but in the Coulomb case $V(r) = a/r$ it is discrete over the interval $-mc^2 < W < mc^2$. If $V(r)$ contains both positive and negative powers of r, the spectrum is continuous over $(-\infty, \infty)$.

My results agree with those of Plesset if $V(r)$ is a polynomial in r, but not if it is a polynomial in $1/r$ of degree greater than the first. It

will be shown that the singularity at the origin then leads to Weyl's limit-circle case. There is an expansion formula containing an arbitrary constant, and for each value of the constant the spectrum is discrete over the range $-mc^2 < W < mc^2$. The solutions are oscillatory as $r \to 0$ for all real values of W, but in Weyl's theory this indicates not a continuous spectrum but the limit-circle case.

2. In my paper (**2**) I considered the theory of eigenfunction expansions associated with a pair of simultaneous equations of the form

$$p(r)\psi_1(r)+q(r)\psi_2(r)+\psi_2'(r)-\lambda\psi_1(r) = 0, \tag{2.1}$$

$$q(r)\psi_1(r)+u(r)\psi_2(r)-\psi_1'(r)-\lambda\psi_2(r) = 0. \tag{2.2}$$

The equations (1.1), (1.2) are the particular case in which

$$p(r) = c^{-1}V(r)-mc, \qquad q(r) = -l/r, \qquad u(r) = c^{-1}V(r)+mc,$$

and $\lambda = W/c$. I begin by making some further remarks on the general case.

Suppose first that $p(r)$, etc., are all absolutely integrable down to $r = 0$. Then there are solutions

$$\theta(r,\lambda) = \begin{bmatrix}\theta_1\\ \theta_2\end{bmatrix}, \qquad \phi(r,\lambda) = \begin{bmatrix}\phi_1\\ \phi_2\end{bmatrix}$$

of (2.1), (2.2) which have given real boundary values at $r = 0$, say

$$\left.\begin{aligned}\theta_1(0,\lambda) &= \cos\alpha, & \theta_2(0,\lambda) &= \sin\alpha\\ \phi_1(0,\lambda) &= -\sin\alpha, & \phi_2(0,\lambda) &= \cos\alpha\end{aligned}\right\}. \tag{2.3}$$

Let $\theta(r,\lambda)+l\phi(r,\lambda)$ be the solution which satisfies a real boundary condition at $r = b$, say

$$\{\theta_1(b,\lambda)+l\phi_1(b,\lambda)\}\cos\beta+\{\theta_2(b,\lambda)+l\phi_2(b,\lambda)\}\sin\beta = 0.$$

This equation determines l. If $b \to \infty$ and $\operatorname{im}\lambda \neq 0$, l tends either to a limit-point or a limit-circle. For example in the limit-point case $l \to m(\lambda)$, and

$$\psi(r,\lambda) = \theta(r,\lambda)+m(\lambda)\phi(r,\lambda) \tag{2.4}$$

is the unique solution which is of integrable squared-modulus to infinity if $\operatorname{im}\lambda \neq 0$.

If $\phi(r,\lambda)$ and $\psi(r,\lambda)$ are two solutions of (2.1), (2.2), then

$$\omega(\lambda) = \psi_1(r,\lambda)\phi_2(r,\lambda)-\psi_2(r,\lambda)\phi_1(r,\lambda) \tag{2.5}$$

is independent of r [cf. (3.4) of (**2**)]. In the above case $\omega(\lambda) = 1$, but it may be convenient to consider functions ϕ and ψ with the same general properties but not normalized in this way, for which $\omega(\lambda)$ is not necessarily 1.

Let $$f(r) = \begin{bmatrix} f_1(r) \\ f_2(r) \end{bmatrix}$$

be a given function of $L^2(0,\infty)$, and let

$$\Phi(r,\lambda) = \frac{\psi(r,\lambda)}{\omega(\lambda)} \int_0^r \phi(y,\lambda) f^T(y)\, dy + \frac{\phi(r,\lambda)}{\omega(\lambda)} \int_r^\infty \psi(y,\lambda) f^T(y)\, dy, \quad (2.6)$$

where in matrix notation $f^T = [f_1, f_2]$ and $\phi f^T = \phi_1 f_1 + \phi_2 f_2$. Thus Φ, like ψ and ϕ, is a two-component function. Consider the integral

$$\int \Phi(r,\lambda)\, d\lambda$$

along $(-R+i\delta, R+i\delta)$, where $\delta > 0$, and round a semicircle above it. Under suitable conditions $\Phi(r,\lambda) \sim f(r)/\lambda$ for large values of λ, and so the integral round the semicircle tends to $\pi i f(r)$ as $R \to \infty$. If $\Phi(r,\lambda)$ is continuous up to the real axis, the expansion formula will therefore be

$$f(r) = -\frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{im} \Phi(r,\lambda)\, d\lambda. \quad (2.7)$$

If $\phi(r,\lambda)$ satisfies (2.3), it is real for all real values of r and λ. Suppose that this is true anyhow. Then, if λ is real,

$$\begin{aligned} \operatorname{im} \frac{\psi_1(r,\lambda)}{\omega(\lambda)} &= \operatorname{im} \frac{\psi_1(r,\lambda)}{\psi_1(r,\lambda)\phi_2(r,\lambda) - \psi_2(r,\lambda)\phi_1(r,\lambda)} \\ &= \frac{1}{2i}\left\{ \frac{\psi_1(r,\lambda)}{\psi_1(r,\lambda)\phi_2(r,\lambda) - \psi_2(r,\lambda)\phi_1(r,\lambda)} - \frac{\bar{\psi}_1(r,\lambda)}{\bar{\psi}_1(r,\lambda)\phi_2(r,\lambda) - \bar{\psi}_2(r,\lambda)\phi_1(r,\lambda)} \right\} \\ &= \frac{\{\psi_2(r,\lambda)\bar{\psi}_1(r,\lambda) - \psi_1(r,\lambda)\bar{\psi}_2(r,\lambda)\}\phi_1(r,\lambda)}{2i|\omega(\lambda)|^2}. \end{aligned} \quad (2.8)$$

If λ is real,

$$\begin{aligned} \frac{d}{dr}\{\psi_1(r,\lambda)\bar{\psi}_2(r,\lambda) - \psi_2(r,\lambda)\bar{\psi}_1(r,\lambda)\} \\ = (q\psi_1 + u\psi_2 - \lambda\psi_2)\bar{\psi}_2 + \psi_1(\lambda\bar{\psi}_1 - p\bar{\psi}_1' - q\bar{\psi}_2) - \\ - (\lambda\psi_1 - p\psi_1' - q\psi_2)\bar{\psi}_1 - \psi_2(q\bar{\psi}_1 + u\bar{\psi}_2' - \lambda\bar{\psi}_2) = 0. \end{aligned}$$

Hence $$\psi_1(r,\lambda)\bar{\psi}_2'(r,\lambda) - \psi_2'(r,\lambda)\bar{\psi}_1(r,\lambda) = 2i\sigma(\lambda) \quad (2.9)$$

say, where $\sigma(\lambda)$ is independent of r. In the case where (2.3), (2.4) hold, $\sigma(\lambda) = -\operatorname{im} m(\lambda)$. Thus

$$\operatorname{im} \frac{\psi_1(r,\lambda)}{\omega(\lambda)} = -\frac{\sigma(\lambda)\phi_1(r,\lambda)}{|\omega(\lambda)|^2}, \quad (2.10)$$

and similarly with suffix 2. The expansion formula is therefore

$$f(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma(\lambda)}{|\omega(\lambda)|^2} \phi(r,\lambda)\, d\lambda \int_0^{\infty} \phi(y,\lambda) f^T(y)\, dy \tag{2.11}$$

(two components on each side).

The spectrum will be continuous over any interval if $\sigma(\lambda)$ is continuous and does not vanish in this interval. This implies that $\omega(\lambda)$ does not vanish either. For, if $\omega(\lambda) = 0$ for a real value of λ, then $\psi_1(r,\lambda)/\psi_2(r,\lambda)$ is real, by (2.5), and so

$$\frac{2i\sigma(\lambda)}{|\psi_2(r,\lambda)|^2} = \frac{\psi_1}{\psi_2} - \frac{\bar{\psi}_1}{\bar{\psi}_2} = 0,$$

which implies that $\sigma(\lambda) = 0$ since $\psi_2(r,\lambda)$ does not vanish identically.

In other cases $\Phi(r,\lambda)$ is regular on part of the real axis except at certain simple poles, and takes real values between the poles. The poles then contribute discrete terms to the expansion formula, and the spectrum is said to be *discrete* over such an interval.

3. If any of $p(r)$, $q(r)$, and $u(r)$ have non-integrable singularities at $r = 0$, the general process for defining solutions of integrable square is as follows. We select some point, say $r = 1$, and define solutions $\theta(r,\lambda)$, $\phi(r,\lambda)$ such that

$$\theta_1(1,\lambda) = 1, \quad \theta_2(1,\lambda) = 0, \quad \phi_1(1,\lambda) = 0, \quad \phi_2(1,\lambda) = 1. \tag{3.1}$$

Let $\theta(r,\lambda)+l\phi(r,\lambda)$ be the solution satisfying a real boundary condition at $r = \rho$, say

$$\{\theta_1(\rho,\lambda)+l\phi_1(\rho,\lambda)\}\cos\alpha+\{\theta_2(\rho,\lambda)+l\phi_2(\rho,\lambda)\}\sin\alpha = 0.$$

As α varies, l describes a circle if $\operatorname{im}\lambda \neq 0$. As $\rho \to 0$, this circle tends either to a limit-point or to a limit-circle. In the former case, let $n(\lambda)$ be the limit-point. Then

$$\chi(r,\lambda) = \theta(r,\lambda)+n(\lambda)\phi(r,\lambda) \tag{3.2}$$

is of integrable squared-modulus if $\operatorname{im}\lambda \neq 0$. The function $\Phi(r,\lambda)$ is defined as before but with $\chi(r,\lambda)$ instead of $\phi(r,\lambda)$, and

$$\omega(\lambda) = m(\lambda)-n(\lambda). \tag{3.3}$$

In the cases to be considered later, $\chi(r,\lambda)$ is real for real values of λ, and hence so is $n(\lambda)$. Thus

$$\operatorname{im}\frac{m(\lambda)}{m(\lambda)-n(\lambda)} = \operatorname{im}\left\{1+\frac{n(\lambda)}{m(\lambda)-n(\lambda)}\right\} = n(\lambda)\operatorname{im}\left\{\frac{1}{m(\lambda)-n(\lambda)}\right\}.$$

The formula (6.2) of (**2**), with interval $(0,\infty)$ instead of $(-\infty,\infty)$, then

gives the expansion formula

$$f(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{im}\left\{\frac{1}{m(\lambda)-n(\lambda)}\right\} \chi(r,\lambda)\, d\lambda \int_0^{\infty} \chi(y,\lambda) f^T(y)\, dy. \tag{3.4}$$

If the solutions are not necessarily normalized by (3.1), we shall have

$$m(\lambda) = \psi_2(1,\lambda)/\psi_1(1,\lambda), \qquad n(\lambda) = \chi_2(1,\lambda)/\chi_1(1,\lambda),$$

and (3.4) will be replaced by

$$f(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} \operatorname{im}\left\{\frac{1}{m(\lambda)-n(\lambda)}\right\} \frac{\chi(r,\lambda)}{\chi_1(1,\lambda)}\, d\lambda \int_0^{\infty} \frac{\chi(y,\lambda)}{\chi_1(1,\lambda)} f^T(y)\, dy. \tag{3.5}$$

If λ is real,

$$\begin{aligned}
&\operatorname{im}\left\{\frac{\psi_2(1,\lambda)}{\psi_1(1,\lambda)} - \frac{\chi_2(1,\lambda)}{\chi_1(1,\lambda)}\right\}^{-1} \\
&\qquad = \frac{1}{2i}\left[\left\{\frac{\psi_2(1,\lambda)}{\psi_1(1,\lambda)} - \frac{\chi_2(1,\lambda)}{\chi_1(1,\lambda)}\right\}^{-1} - \left\{\frac{\overline{\psi_2(1,\lambda)}}{\overline{\psi_1(1,\lambda)}} - \frac{\chi_2(1,\lambda)}{\chi_1(1,\lambda)}\right\}^{-1}\right] \\
&\qquad = \frac{\{\psi_1(1,\lambda)\overline{\psi_2(1,\lambda)} - \psi_2(1,\lambda)\overline{\psi_1(1,\lambda)}\}\chi_1^2(1,\lambda)}{2i|\psi_2(1,\lambda)\chi_1(1,\lambda) - \psi_1(1,\lambda)\chi_2(1,\lambda)|^2} = \frac{\sigma(\lambda)\chi_1^2(1,\lambda)}{|\omega(\lambda)|^2}
\end{aligned}$$

if χ now replaces ϕ in the definition of $\omega(\lambda)$. Hence (3.5) gives

$$f(r) = \frac{1}{\pi} \int_{-\infty}^{\infty} \frac{\sigma(\lambda)}{|\omega(\lambda)|^2} \chi(r,\lambda)\, d\lambda \int_0^{\infty} \chi(y,\lambda) f^T(y)\, dy. \tag{3.6}$$

As before, the spectrum will be continuous over any interval where $\sigma(\lambda)$ is continuous and not zero.

There will clearly be a general expansion formula, including all the above cases, of the form

$$f(r) = \int_{-\infty}^{\infty} \chi(r,\lambda) F(\lambda)\, d\rho(\lambda), \tag{3.7}$$

where

$$F(\lambda) = \int_0^{\infty} \chi(y,\lambda) f^T(y)\, dy \tag{3.8}$$

and $\rho(\lambda)$ is a non-decreasing function of λ. The corresponding Parseval formula will be

$$\int_0^{\infty} |f(r)|^2\, dr = \int_{-\infty}^{\infty} \{F(\lambda)\}^2\, d\rho(\lambda), \tag{3.9}$$

where $|f|^2 = f_1^2 + f_2^2$. For example, in (3.6)

$$\rho'(\lambda) = \pi^{-1}\sigma(\lambda)|\omega(\lambda)|^{-2},$$

while in the case of a discrete spectrum $\rho(\lambda)$ is a step-function.

4. Existence of solutions of (2.1), (2.2)

If $p(r)$, $q(r)$, and $u(r)$ are integrable in the neighbourhood of $r = 0$, a solution can be defined as follows. The equations (2.1), (2.2) with ψ replaced by ϕ are equivalent to the integral equations

$$\phi_1(r)-\phi_1(0) = \int_0^r (q\phi_1+u\phi_2-\lambda\phi_2)\,dt, \tag{4.1}$$

$$\phi_2(r)-\phi_2(0) = \int_0^r (-p\phi_1-q\phi_2+\lambda\phi_1)\,dt. \tag{4.2}$$

Let A and B be given constants, let $\phi_1^{(0)}(r) = A$, $\phi_2^{(0)}(r) = B$, and, for $n \geqslant 0$,

$$\phi_1^{(n+1)}(r)-A = \int_0^r (q\phi_1^{(n)}+u\phi_2^{(n)}-\lambda\phi_2^{(n)})\,dt,$$

$$\phi_2^{(n+1)}(r)-B = \int_0^r (-p\phi_1^{(n)}-q\phi_2^{(n)}+\lambda\phi_1^{(n)})\,dt.$$

Let $M = \max(|A|, |B|)$ and

$$J(r) = \int_0^r (|p(t)|+|q(t)|+|u(t)|+|\lambda|)\,dt.$$

Then
$$|\phi_\nu^{(1)}(r)-\phi_\nu^{(0)}(r)| \leqslant MJ(r) \quad (\nu = 1, 2),$$

$$|\phi_1^{(2)}(r)-\phi_1^{(1)}(r)| = \left|\int_0^r \{q(\phi_1^{(1)}-\phi_1^{(0)})+(u-\lambda)(\phi_2^{(1)}-\phi_2^{(0)})\}\,dt\right|$$

$$\leqslant M\int_0^r (|q|+|u|+|\lambda|)J(t)\,dt$$

$$\leqslant M\int_0^r J'(t)J(t)\,dt = \tfrac{1}{2}MJ^2(r),$$

and similarly for $\phi_2^{(2)}$. So, generally, we get

$$|\phi_\nu^{(n)}(r)-\phi_\nu^{(n-1)}(r)| \leqslant \frac{MJ^n(r)}{n!}.$$

The iterative process therefore converges and defines solutions with given boundary values.

Next let $p(r)$ and $u(r)$ be integrable but let $q(r) = -l/r$, where $l > \frac{1}{2}$. Then (2.1), (2.2) with χ instead of ϕ are

$$\frac{d}{dr}\{r^l\chi_1(r)\} = r^l\{u(r)-\lambda\}\chi_2(r), \tag{4.3}$$

$$\frac{d}{dr}\{r^{-l}\chi_2(r)\} = r^{-l}\{\lambda-p(r)\}\chi_1(r). \tag{4.4}$$

These equations will be satisfied if

$$r^l\chi_1(r) = \int_0^r t^l\{u(t)-\lambda\}\chi_2(t)\,dt, \tag{4.5}$$

$$r^{-l}\chi_2(r) = 1+\int_0^r t^{-l}\{\lambda-p(t)\}\chi_1(t)\,dt. \tag{4.6}$$

These can be solved by iteration by taking $\chi_2^{(0)}(r) = r^l$ and

$$r^l\chi_1^{(n)}(r) = \int_0^r t^l\{u(t)-\lambda\}\chi_2^{(n)}(t)\,dt,$$

$$r^{-l}\chi_2^{(n+1)}(r) = 1+\int_0^r t^{-l}\{\lambda-p(t)\}\chi_1^{(n)}(t)\,dt.$$

We thus obtain a solution such that $\chi_1 = O(r^{l+1})$, $\chi_2 = O(r^l)$ as $r \to 0$. It is the only solution of integrable square (apart from constant factors). For, if there were another, say ψ, and

$$\omega(\lambda) = \psi_1(r,\lambda)\chi_2(r,\lambda)-\psi_2(r,\lambda)\chi_1(r,\lambda),$$

then as $r \to 0$,

$$\begin{aligned} r\omega(\lambda) &= \int_0^r \{\psi_1(t,\lambda)\chi_2(t,\lambda)-\psi_2(t,\lambda)\chi_1(t,\lambda)\}\,dt \\ &= O\Big(r^l\int_0^r |\psi_1|\,dt\Big)+O\Big(r^{l+1}\int_0^r |\psi_2|\,dt\Big) = O(r^{l+\frac{1}{2}}) \end{aligned}$$

by the Schwarz inequality. Hence $\omega(\lambda) = 0$.

There is a similar situation if also

$$p(r) = p_0(r)-\alpha/r, \qquad u(r) = u_0(r)-\alpha/r,$$

where $p_0(r)$ and $u_0(r)$ are integrable and $|\alpha| < l$. The equations (2.1), (2.2) are then

$$\chi_2'-\alpha\chi_1/r-l\chi_2/r = (\lambda-p_0)\chi_1, \tag{4.7}$$

$$\chi_1'+l\chi_1/r+\alpha\chi_2/r = (u_0-\lambda)\chi_2. \tag{4.8}$$

Hence

$$\chi_2'+k\chi_1'+(kl-\alpha)\chi_1/r+(k\alpha-l)\chi_2/r = (\lambda-p_0)\chi_1+k(u_0-\lambda)\chi_2.$$

The left-hand side depends on $\chi_2+k\chi_1$ only if $kl-\alpha = k(k\alpha-l)$, i.e.

$$k = \frac{l}{\alpha}\pm\sqrt{\Big(\frac{l^2}{\alpha^2}-1\Big)} = \frac{k_1}{k_2}, \quad \text{say.}$$

Hence

$$\chi_2'+k_1\chi_1'+\sqrt{(l^2-\alpha^2)}(\chi_2+k_1\chi_1)/r = (\lambda-p_0)\chi_1+k_1(u_0-\lambda)\chi_2,$$

$$\chi_2'+k_2\chi_1'-\sqrt{(l^2-\alpha^2)}(\chi_2+k_2\chi_1)/r = (\lambda-p_0)\chi_1+k_2(u_0-\lambda)\chi_2.$$

These are of the same form as before with l replaced by $\sqrt{(l^2-\alpha^2)}$, and

similar results hold. In all these cases $\chi_1(r,\lambda)$ and $\chi_2(r,\lambda)$ are integral functions of λ for each value of r, and are real when λ is real.

5. We now return to the particular case of the equations (1.1), (1.2). Let $V(r)$ be twice differentiable. Expressing ψ_2 and ψ_2' in terms of ψ_1 by (1.2), and substituting in (1.1), we obtain

$$(W-V+mc^2)\psi_1+\frac{c^2}{W-V-mc^2}\left(\psi_1''-\frac{l^2+l}{r^2}\psi_1\right)+ \\ +\frac{c^2V'}{(W-V-mc^2)^2}\left(\psi_1'+\frac{l}{r}\psi_1\right)=0. \quad (5.1)$$

Let
$$\zeta(r)=(W-V-mc^2)^{-\frac{1}{2}}\psi_1, \quad (5.2)$$

where the square root is real and positive if W is real and greater than $V+mc^2$. Then (5.1) gives

$$\zeta''=\left\{\frac{l^2+l}{r^2}-\frac{V'}{W-V-mc^2}\frac{l}{r}-\frac{(W-V)^2-m^2c^4}{c^2}+ \right. \\ \left. +\frac{3V'^2}{4(W-V-mc^2)^2}+\frac{V''}{2(W-V-mc^2)}\right\}\zeta. \quad (5.3)$$

This can be written as
$$\zeta''+\{s^2-Q(r)\}\zeta=0, \quad (5.4)$$

where
$$Q(r)=\frac{l^2+l}{r^2}-\frac{V'}{W-V-mc^2}\frac{l}{r}+\frac{2WV-V^2}{c^2}+ \\ +\frac{3V'^2}{4(W-V-mc^2)^2}+\frac{V''}{2(W-V-mc^2)}$$

and
$$s=(W^2-m^2c^4)^{\frac{1}{2}}/c,$$

where the square root is real and positive if $W>mc^2$, and other points such that $\operatorname{im} W\geqslant 0$ are reached by paths in the upper half-plane. The intervals

$$(-\infty,-mc^2), \quad (-mc^2,0), \quad (0,mc^2), \quad (mc^2,\infty)$$

of the real W-axis correspond to the intervals

$$(-\infty,0), \quad (0,imc), \quad (imc,0), \quad (0,\infty)$$

in the s-plane. The upper half of the W-plane corresponds to the upper half of the s-plane cut along $(0,imc)$.

Now suppose that $V(r)$, $V'(r)/V(r)$ and $V''(r)/V(r)$ tend to 0 as $r\to\infty$, and that
$$V(r), \quad \{V'(r)/V(r)\}^2, \quad V''(r)/V(r)$$

are absolutely integrable to infinity. Then $Q(r)$ tends to 0 as $r\to\infty$ and is absolutely integrable over (a,∞) if $|V(r)|<|W-mc^2|$ for $r\geqslant a$.

The equation (5.4) is formally equivalent to the integral equation

$$\zeta(r) = e^{isr} + \frac{1}{s}\int_r^\infty \sin\{s(y-r)\}Q(y)\zeta(y)\,dy, \tag{5.5}$$

or, if $f(r) = e^{-isr}\zeta(r)$, to

$$f(r) = 1 + \frac{1}{2is}\int_r^\infty (e^{2is(y-r)}-1)Q(y)f(y)\,dy. \tag{5.6}$$

If $\operatorname{im} s \geqslant 0$, this can be solved by the usual process of iteration. The solution is bounded for $r \geqslant r_0$, $|s| \geqslant s_0$, and $f(r) \to 1$ as $r \to \infty$. Thus

$$\zeta(r) \sim e^{isr} \tag{5.7}$$

and so $\zeta(r)$ is exponentially small as $r \to \infty$ if $\operatorname{im} s > 0$. Also, by differentiating (5.5), we have

$$\zeta'(r) = ise^{isr} - \int_r^\infty \cos\{s(y-r)\}Q(y)\zeta(y)\,dy,$$

which gives, as $r \to \infty$, $\qquad \zeta'(r) \sim ise^{isr}$. $\tag{5.8}$

Since $V(r) \to 0$, $V'(r) \to 0$, the corresponding function $\psi_1(r)$ satisfies

$$\psi_1(r) \sim (W-mc^2)^{\frac{1}{2}}e^{isr}, \tag{5.9}$$

$$\psi_1'(r) = (W-V-mc^2)^{\frac{1}{2}}\zeta'(r) - \tfrac{1}{2}V'(W-V-mc^2)^{-\frac{1}{2}}\zeta(r) \sim (W-mc^2)^{\frac{1}{2}}ise^{isr}. \tag{5.10}$$

Hence, by (1.2), $\qquad \psi_2(r) \sim -ics(W-mc^2)^{-\frac{1}{2}}e^{isr}$. $\tag{5.11}$

6. Let $V(r)$ satisfy the conditions of § 5 as $r \to \infty$, and let it be integrable down to $r = 0$, or let $V(r) = -ar^{-1} + V_0(r)$ where $V_0(r)$ is integrable.

If W is real and greater than mc^2, s is real, and the conjugates of (5.9) and (5.11) are

$$\bar{\psi}_1(r) \sim (W-mc^2)^{\frac{1}{2}}e^{-isr}, \qquad \bar{\psi}_2(r) \sim ics(W-mc^2)^{-\frac{1}{2}}e^{-isr}.$$

Hence, as $r \to \infty$, $\qquad \psi_1(r)\bar{\psi}_2(r) - \psi_2(r)\bar{\psi}_1(r) \sim 2ics,$

and so the two sides are equal since the left-hand side is independent of r. Thus, in the notation of §§ 2–3,

$$\sigma(\lambda) = \sigma(W/c) = cs. \tag{6.1}$$

Since this is not zero, it follows that the spectrum is continuous over the range $W > mc^2$. Similarly it is continuous over $W < -mc^2$.

On the other hand, if $-mc^2 < W < mc^2$, s is purely imaginary, $\operatorname{im} s > 0$. The functions ψ_1 and ψ_2 tend to 0 as $r \to \infty$, and so $\sigma(\lambda) = 0$.

Let $W = W_1 + iW_2$ where

$$\delta \leqslant W_1 \leqslant mc^2 - \delta \quad (\delta > 0), \qquad -\epsilon \leqslant W_2 \leqslant \epsilon.$$

As $\epsilon \to 0$, the corresponding region in the s-plane tends to the interval

$$ic^{-1}(2mc^2\delta-\delta^2)^{\frac{1}{2}}, \qquad ic^{-1}(m^2c^4-\delta^2)^{\frac{1}{2}}.$$

Hence, if ϵ is small enough, the s-region is in the upper half-plane. In such a region the process of iteration by which (5.6) is solved converges uniformly, and so $f(r) = f(r, s)$ is an analytic function of s. Following the analysis through we see that $\omega(\lambda)$ is analytic in such a region, and so $\Phi(r, \lambda)$ is meromorphic. It follows from the general theory that its singularities are simple poles on the real axis. The spectrum is therefore discrete over the range $0 < W < mc^2$. Similarly it is discrete over $-mc^2 < W < 0$.

7. The case $V(r) \to 0$, $\int^{\infty} \{V(r)\}^{-1}\,dr$ divergent

In this case we have to make a transformation in order to approximate to solutions of (5.3). Let

$$K(r) = \{W-V(r)\}^2/c^2-m^2c^2, \tag{7.1}$$

$$\xi(r_1, r) = \int_{r_1}^{r} \{K(u)\}^{\frac{1}{2}}\,du, \qquad \eta(r) = \{K(r)\}^{\frac{1}{4}}\zeta(r), \tag{7.2}$$

where $K^{\frac{1}{2}}(r)$ and $K^{\frac{1}{4}}(r)$ are real and positive if W is real and greater than $V(r)+mc^2$, and otherwise are determined by variation along a path in the upper half-plane. The equation (5.3) transforms into

$$\frac{d^2\eta}{d\xi^2} = \{P(r)-1\}\eta, \tag{7.3}$$

where

$$P(r) = \frac{l^2+l}{r^2K(r)} - \frac{lV'(r)}{\{W-V(r)-mc^2\}rK(r)} + \frac{3V'^2(r)}{4\{W-V(r)-mc^2\}^2K(r)} + \\ + \frac{V''(r)}{2\{W-V(r)-mc^2\}K(r)} + \frac{K''(r)}{4K^2(r)} - \frac{5K'^2(r)}{16K^3(r)}.$$

As $r \to \infty$, , $K(r) \to W^2/c^2-m^2c^2$,

and, if $W \neq \pm mc^2$,

$$K'(r) = O\{V'(r)\}, \qquad K''(r) = O\{V'^2(r)\}+O(V''(r)\}.$$

Hence $P(r)\{K(r)\}^{\frac{1}{2}}$ is absolutely integrable to infinity if $V'^2(r)$ and $|V''(r)|$ are integrable.

A solution of (7.3) is obtained by solving the integral equation

$$\eta(r) = e^{i\xi(r_1,r)} + \int_{r}^{\infty} \sin\xi(r, u)P(u)\{K(u)\}^{\frac{1}{2}}\eta(u)\,du \tag{7.4}$$

by iteration. As $r \to \infty$, $\eta(r) \sim e^{i\xi(r_1,r)}$ (7.5)

and $$\xi(r_1, r) \sim rs \tag{7.6}$$

where s is defined as before. Thus $\eta(r)$ is exponentially small for large r if im $W > 0$.

By (7.2), (7.5), and (5.2), as $r \to \infty$,

$$\psi_1(r) \sim c^{\frac{1}{2}}\left(\frac{W-mc^2}{W+mc^2}\right)^{\frac{1}{4}} e^{i\xi(r_1,r)}. \tag{7.7}$$

It is seen by differentiating (7.4) that the results of formally differentiating these asymptotic formulae are valid. Thus

$$\psi_1'(r) \sim c^{\frac{1}{2}}\left(\frac{W-mc^2}{W+mc^2}\right)^{\frac{1}{4}} i\{K(r)\}^{\frac{1}{2}} e^{i\xi(r_1,r)} \sim \frac{i}{c^{\frac{1}{2}}}\left(\frac{W-mc^2}{W+mc^2}\right)^{\frac{1}{4}} (W^2-m^2c^4)^{\frac{1}{2}} e^{i\xi(r_1,r)},$$

and so, by (1.2), $$\psi_2(r) \sim -ic^{\frac{1}{2}}\left(\frac{W+mc^2}{W-mc^2}\right)^{\frac{1}{4}} e^{i\xi(r_1,r)}. \tag{7.8}$$

If $W > mc^2$, $\xi(r_1, r)$ is real if r_1 and r are large enough, and, from (7.7) and (7.8) and their conjugates, we obtain, as $r \to \infty$,

$$\psi_1(r)\bar{\psi}_2(r) - \bar{\psi}_1(r)\psi_2(r) \sim 2ic.$$

It follows as before that $\sigma(W/c) = c$, and we conclude that the spectrum is continuous over the range $W > mc^2$, and similarly over $W < -mc^2$.

If $-mc^2 < W < mc^2$, $\xi(r_1, r)$ is purely imaginary if r_1 and r are large enough, and we find as before that $\sigma(W/c) = 0$. The same argument as before shows that the spectrum is discrete over the interval

$$-mc^2 < W < mc^2.$$

8. The case $V(r) \to \infty$ as $r \to \infty$

We make the same transformation as before, and now, according to our conventions, $\arg\{K(r)\}^{\frac{1}{2}} \to \pi$ as $r \to \infty$. Hence, as $r \to \infty$,

$$\xi(r_1, r) \sim -\frac{1}{c}\int_{r_1}^{r} \{V(u) - W\}\, du = -\frac{1}{c}\int_{r_1}^{r} V(u)\, du + \frac{W(r-r_1)}{c}.$$

The imaginary part of this tends to infinity with r if im $W > 0$. The solution which is of integrable squared-modulus up to infinity is therefore obtained as before provided that $P(r)\{K(r)\}^{\frac{1}{2}}$ is integrable. This is so if $V'^2(r)/V^3(r)$ and $|V''(r)/V^2(r)|$ are integrable to infinity.

We now have, as $r \to \infty$,

$$\zeta(r) \sim \{K(r)\}^{-\frac{1}{4}} e^{i\xi(r_1,r)} \sim -ic^{\frac{1}{2}}\{V(r)\}^{-\frac{1}{4}} e^{i\xi(r_1,r)},$$

$$\psi_1(r) \sim c^{\frac{1}{2}} e^{i\xi(r_1,r)}, \tag{8.1}$$

$$\psi_1'(r) \sim c^{\frac{1}{2}} i\{K(r)\}^{\frac{1}{2}} e^{i\xi(r_1,r)} \sim -ic^{-\frac{1}{2}} V(r) e^{i\xi(r_1,r)},$$

$$\psi_2(r) \sim c\psi_1'(r)/V(r) \sim -ic^{\frac{1}{2}} e^{i\xi(r_1,r)}. \tag{8.2}$$

If W is real and r_1 and r are large enough, $\xi(r_1, r)$ is real, and we obtain

$$\psi_1(r)\bar{\psi}_2(r)-\psi_2(r)\bar{\psi}_1(r) \sim 2ic.$$

Hence $\sigma(W/c) = c$ for all values of W. It follows that the spectrum is continuous over the whole range $(-\infty, \infty)$.

9. The case $V(r) \to \infty$ as $r \to 0$

If the singular part of $V(r)$ is of the form $-a/r$, the previous analysis applies, but otherwise it is convenient to transform the singularity to infinity and use the method of §§ 5–7. Let

$$r = y^{-1}, \qquad \tau(y) = y\zeta(y^{-1}).$$

Then

$$\frac{d^2\zeta}{dr^2} = y^4\frac{d^2\zeta}{dy^2}+2y^3\frac{d\zeta}{dy} = y^3\frac{d^2\tau}{dy^2}.$$

Hence on dividing (5.3) by y^3 we obtain

$$\frac{d^2\tau}{dy^2} = \left\{\frac{l^2+l}{y^2}+\frac{l\,dV/dy}{(W-V-mc^2)y}-\frac{(W-V)^2-m^2c^4}{c^2y^4}+\right.$$
$$\left.+\frac{3(dV/dy)^2}{4(W-V-mc^2)^2}+\frac{\frac{1}{2}d^2V/dy^2+y^{-1}dV/dy}{W-V-mc^2}\right\}\tau(y). \quad (9.1)$$

The simplest case is that in which $V = ar^{-2} = ay^2$. Then

$$\frac{d^2\tau}{dy^2}+\left\{\frac{a^2}{c^2}-M(y)\right\}\tau(y) = 0, \quad (9.2)$$

where $M(y) = O(y^{-2})$ as $y \to \infty$.

Let $\theta(r, W)$ and $\phi(r, W)$ be solutions of the original equations with given values at a point r_0 such that

$$|W-mc^2| < \min_{r<r_0}|V(r)|,$$

so that $W-V(r)-mc^2$ does not vanish for $r < r_0$. Let

$$\zeta(r) = \{W-V(r)-mc^2\}^{-\frac{1}{2}}\theta_1(r, W).$$

Then the corresponding $\tau(y)$ and its derivative have given values, which are analytic functions of W, at $y = y_0 = 1/r_0$.

Now $\tau(y)$ satisfies the integral equation

$$\tau(y) = \cos\left\{\frac{a(y-y_0)}{c}\right\}\tau(y_0)+\frac{c}{a}\sin\left\{\frac{a(y-y_0)}{c}\right\}\tau'(y_0)-$$
$$-\int_{y_0}^{y}\sin\left\{\frac{a(y-z)}{c}\right\}M(z)\tau(z)\,dz, \quad (9.3)$$

and this gives, as $y \to \infty$,

$$\tau(y) \sim \cos\left\{\frac{a(y-y_0)}{c}\right\}\tau(y_0)+\frac{c}{a}\sin\left\{\frac{a(y-y_0)}{c}\right\}\tau'(y_0)-$$

$$-\int_{y_0}^{\infty}\sin\left\{\frac{a(y-z)}{c}\right\}M(z)\tau(z)\,dz = P\cos(ay/c)+Q\sin(ay/c),$$

say, where P and Q are analytic functions of W, and are real when W is real. Hence, as $r \to 0$,

$$\theta_1(r, W) \sim ia^{\frac{1}{2}}\left(P\cos\frac{a}{cr}+Q\sin\frac{a}{cr}\right). \tag{9.4}$$

As in previous cases the results of formal differentiation are correct, so that

$$\theta_1'(r, W) \sim \frac{ia^{\frac{3}{2}}}{cr^2}\left(P\sin\frac{a}{cr}-Q\cos\frac{a}{cr}\right).$$

Hence

$$\theta_2(r, W) \sim ia^{\frac{1}{2}}\left(P\sin\frac{a}{cr}-Q\cos\frac{a}{cr}\right). \tag{9.5}$$

Similarly, there are analytic functions $R = R(W)$, $S = S(W)$ such that

$$\phi_1(r, W) \sim ia^{\frac{1}{2}}\left(R\cos\frac{a}{cr}+S\sin\frac{a}{cr}\right), \tag{9.6}$$

$$\phi_2(r, W) \sim ia^{\frac{1}{2}}\left(R\sin\frac{a}{cr}-S\cos\frac{a}{cr}\right). \tag{9.7}$$

Let $\theta(r, W)+l\phi(r, W)$ be the solution which satisfies a real boundary condition at $r = \rho$, say

$$\{\theta_1(\rho, W)+l\phi_1(\rho, W)\}\cos\alpha+\{\theta_2(\rho, W)+l\phi_2(\rho, W)\}\sin\alpha = 0.$$

Then

$$l = -\frac{\theta_1(\rho, W)\cos\alpha+\theta_2(\rho, W)\sin\alpha}{\phi_1(\rho, W)\cos\alpha+\phi_2(\rho, W)\sin\alpha}$$

$$\sim -\frac{P\cos\{a/(c\rho)-\alpha\}+Q\sin\{a/(c\rho)-\alpha\}}{R\cos\{a/(c\rho)-\alpha\}+S\sin\{a/(c\rho)-\alpha\}}.$$

Let $\alpha = a/(c\rho)-\beta$, where β is independent of ρ. Then, as $\rho \to 0$,

$$l \to m = -\frac{P\cos\beta+Q\sin\beta}{R\cos\beta+S\sin\beta}.$$

If W is not real, P, Q, R and S are not real, and, as β varies, m describes a circle, which is therefore the limit-circle. For any given value of β, $m = m(W)$ is an analytic function of W, and it is real when W is real. It follows that, for any given value of β, a spectrum is obtained which is of the same character, discrete or continuous, as if there were no singularity at $r = 0$.

10. To obtain asymptotic formulae for solutions of (9.1) in more general cases we have to make a transformation of the type used in § 7, with

$$K(y) = \{(W-V)^2 - m^2c^4\}/(c^4y^4).$$

The result will be of the form

$$\tau(y) \sim \frac{c^{\frac{1}{2}}y}{V^{\frac{1}{2}}} \exp\left(i \int_{y_1}^{y} \frac{[\{W-V\}^2 - m^2c^4]^{\frac{1}{2}}}{ct^2}\, dt\right).$$

Hence, as $r \to 0$,

$$\theta_1(r, W) \sim P\cos\{Z(r)\} + Q\sin\{Z(r)\},$$

where

$$Z = \frac{1}{c}\int_{y_1}^{y} \frac{V}{t^2}\, dt = \frac{1}{c}\int_{r}^{r_1} V(u)\, du,$$

and P and Q are analytic functions of W. As $r \to 0$, $Z(r) \to \infty$, and analysis similar to that of § 9 shows that we are in the limit-circle case, with a spectrum of the same form as if there were no singularity at $r = 0$. It will be seen that these asymptotic formulae agree with Plesset's, but the conclusion drawn from them is different.

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